# Math 630 Linear Algebra
## Final Exam

## Problem 1

1)
$$\begin{pmatrix} 3 & 3 & 0 & b_1 \\ 3 & 10 & 7 & b_2 \\ 0 & 7 & 8 & b_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 3 & 0 & b_1 \\ 0 & 7 & 7 & b_2-b_1 \\ 0 & 7 & 8 & b_3 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 3 & 3 & 0 & b_1 \\ 0 & 7 & 7 & b_2-b_1 \\ 0 & 0 & 1 & b_3-b_2+b_1 \end{pmatrix}$$

Thus,
$$u = \frac{b_1}{3} - \left(\frac{b_2-b_1}{7} - b_3+b_2-b_1\right)$$
$$v = \frac{b_2-b_1}{7} - (b_3-b_2+b_1)$$
$$w = b_3-b_2+b_1$$

2) $\underline{A} - 2\underline{I}$

$$\begin{pmatrix} 1 & 3 & 0 \\ 3 & 8 & 7 \\ 0 & 7 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 7 \\ 0 & 7 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} \underline{1} & 3 & 0 \\ 0 & \underline{\underline{-1}} & 7 \\ 0 & 0 & \underline{\underline{55}} \end{pmatrix}$$

One eigenvalue $< 2$

## Problem 2

$$A^{-1} + B^{-1} = B^{-1} + A^{-1} = A^{-1}(A+B)B^{-1}$$
$$(A^{-1}+B^{-1})^{-1} = (A^{-1}(A+B)B^{-1})^{-1} = B(A+B)^{-1}A$$

## Problem 3

1) $$\underline{x}_{\text{particular}} = \begin{bmatrix} a-3b \\ 0 \\ b \\ 0 \end{bmatrix}, \text{ by assuming } \underbrace{x_2 = x_4 = 0}_{\text{free variables}}$$

$x_1, x_3 \leftarrow$ pivot variables

Essentially from solving $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

Null(A):

$\checkmark$: F components

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Nullspace vectors $\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$ ✓: $-\underline{E}$

$$\underline{X} = \underline{X}_{\text{particular}} + x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

2) $\underline{A} \neq \underline{B}$ in general. Example: $\underline{A} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $\underline{B} = \begin{bmatrix}2&0\\0&1\end{bmatrix}$

## Problem 4

The plane perpendicular to $(1,1,0)$ is
$x + y = 0$
The plane perpendicular to $(0,1,1)$ is
$y + z = 0$
The intersection is formed from the solution

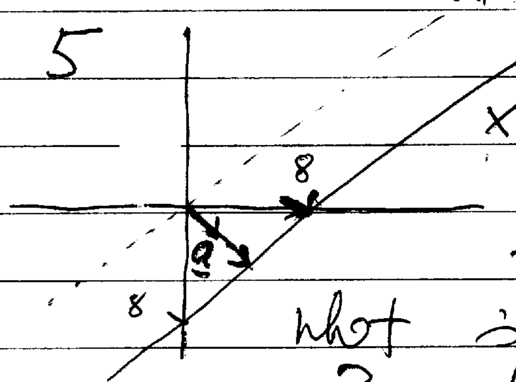
$$\begin{cases} x+y=0 \\ y+z=0 \end{cases} \Longrightarrow \begin{pmatrix} z\\-z\\z\end{pmatrix} = z\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Notice that $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ is the Nullspace vector of

$\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix}$! why? $\begin{pmatrix}1\\-1\\1\end{pmatrix} \perp \begin{pmatrix}1\\1\\0\end{pmatrix}$; $\begin{pmatrix}1\\-1\\1\end{pmatrix} \perp \begin{pmatrix}0\\1\\1\end{pmatrix}$!

$x_1 - x_2 = 0$

## Problem 5




$x_1 - x_2 = 8$

Two ways of doing this.
A). choose line $x_1 - x_2 = 0$, to project the vector $\begin{pmatrix}8\\0\end{pmatrix}$, what is the vector representing line $x_1 - x_2 = 0$? $\begin{pmatrix}1\\1\end{pmatrix}$! (also Nullspace, why?)

$$\underline{P} = \underline{A}(\underline{A}^T\underline{A})^{-1}\underline{A}^T = \begin{pmatrix}1\\1\end{pmatrix}\left(\frac{1}{2}\right)\begin{pmatrix}1&1\end{pmatrix} = \begin{pmatrix}1/2&1/2\\1/2&1/2\end{pmatrix}$$

$$\underline{a} = (\underline{I} - \underline{P})\begin{pmatrix} 8 \\ 0 \end{pmatrix} \qquad \text{(why?)}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}. \quad \|a\| = 4\sqrt{2}.$$

B) recognizing $\underline{a}$ direction and use it as projection vector (the vector setting up $\underline{P}$ matrix

$$\underline{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \underline{P} = \underline{a}(\underline{a}^T\underline{a})^{-1}\underline{a}^T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\underline{b} = \underline{P}\begin{pmatrix} 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}, \qquad \|\underline{b}\| = 4\sqrt{2}$$

Problem 6

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ to } \begin{vmatrix} a-mc & b-md \\ c-la & d-lb \end{vmatrix}$$

First of all $\begin{vmatrix} a-mc & b-md \\ c-la & d-lb \end{vmatrix}$

$$= \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} - m\begin{vmatrix} c & d \\ c-la & d-lb \end{vmatrix}$$

Then

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - m\begin{vmatrix} c & d \\ -la & -lb \end{vmatrix} = \underbrace{(1-ml)}_{\text{be careful!}}\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Problem 7

$$\det \underbrace{\begin{bmatrix} \underset{m\times m}{\underline{O}} & \underset{m\times n}{\underline{A}} \\ \underset{n\times m}{-\underline{B}} & \underset{n\times n}{\underline{I}} \end{bmatrix}}_{(m+n)\times(m+n)} \underbrace{\begin{bmatrix} \underset{m\times m}{\underline{I}} & \underset{m\times n}{\underline{O}} \\ \underset{n\times m}{\underline{B}} & \underset{n\times n}{\underline{I}} \end{bmatrix}}_{(m+n)\times(m\times n)} = \det\begin{bmatrix} \underline{O} & \underline{A} \\ -\underline{B} & \underline{I} \end{bmatrix} \det\begin{bmatrix} \underline{I} & \underline{O} \\ \underline{B} & \underline{I} \end{bmatrix}.$$

Note $\det \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{B} & \underline{I} \end{bmatrix} = 1$

$$\begin{bmatrix} \underline{0} & \underline{A} \\ -\underline{B} & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{B} & \underline{I} \end{bmatrix} = \begin{array}{cc} m\times m & m\times n \\ \left[\begin{matrix} \underline{A}\,\underline{B} & \underline{A} \\ \underline{0} & \underline{I} \end{matrix}\right] \\ n\times m & n\times n \end{array}$$

$$\det \begin{bmatrix} \underline{A}\,\underline{B} & \underline{A} \\ \underline{0} & \underline{I} \end{bmatrix} = \det(\underline{A}\,\underline{B})$$

(Don't use $\det(\underline{A}\,\underline{B}) = \det \underline{A} \det \underline{B}$ )
$\underline{A}$, $\underline{B}$ in general are rectangular!

Thus $\det \begin{bmatrix} \underline{0} & \underline{A} \\ -\underline{B} & \underline{I} \end{bmatrix} = \det(\underline{A}\,\underline{B})$

Example $\underline{A} = [1 \quad 0]$, $\underline{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$m<n$

$\det(\underline{A}\,\underline{B}) = 1$

$$\begin{bmatrix} \underline{0} & \underline{A} \\ -\underline{B} & \underline{I} \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline -1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad \det \begin{bmatrix} \underline{0} & \underline{A} \\ -\underline{B} & \underline{I} \end{bmatrix} = 1 = \det(\underline{A}\,\underline{B})$$

Problem 8

$$\det\left( \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0 \Rightarrow \lambda_1 = 0, \ \lambda_2 = 1$$

$\lambda_1 = 0$, $\underline{s}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\lambda_2 = 1$, $\underline{s}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{S} = (\underline{s}_1 \ \underline{s}_2)$

$$\underline{u} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} e^{0t} & 0 \\ 0 & e^t \end{pmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\substack{\text{Nullspace} \\ \text{fixed}}} e^{\overset{\equiv 1}{0t}} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\substack{\text{Column space} \\ \text{exponentially increase}}} e^{t} \cdot 4$$

Problem 9

1) $\begin{vmatrix} a-\lambda & b \\ 1-a & 1-b-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = a-b, \quad \lambda_2 = 1$

$\lambda_1 = a-b, \quad \underline{s}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad \lambda_2 = 1, \quad \underline{s}_2 = \begin{bmatrix} -b \\ a-1 \end{bmatrix}, \quad \underline{S} = (\underline{s}_1 \ \underline{s}_2)$

$$\underline{U}_k = \begin{bmatrix} 1 & -b \\ -1 & a-1 \end{bmatrix} \begin{bmatrix} (a-b)^k & 0 \\ 0 & 1^k \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -b \\ -1 & a-1 \end{bmatrix}^{-1} \underline{u}_0}_{\text{leave it there, since } \underline{u}_0 \text{ is unknown}}$$

2) If $a \neq b$, $(a-b)^k \to 0$ as $k \to \infty$

$$\underline{U}_k \longrightarrow \begin{bmatrix} -b \\ a-1 \end{bmatrix} C, \quad C = \underbrace{\text{second row of}}_{= \frac{2}{a+b}} \underbrace{\begin{bmatrix} 1 & -b \\ -1 & a-1 \end{bmatrix}^{-1}}_{\left[\frac{1}{a-1-b} \ \frac{1}{a-1-b}\right]} \cdot \underline{u}_0$$

Problem 10

$$\underline{K} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}, \quad \underline{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(\underline{K} - \omega^2 \underline{M}) = 0$$

$$\Rightarrow \omega_1 = 1, \quad \omega_2 = 3$$
$$(\lambda_1 = 1) \quad (\lambda_2 = 9)$$

$$\underline{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \underline{q}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\underline{U} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} (c_1 \cos t + c_2 \sin t) + \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} (d_1 \cos 3t + d_2 \sin 3t)$$

How to find $c_1, c_2; d_1, d_2$ ?

Note that $\quad U = Y_1 \underline{q}_1 + \underline{Y_2} \underline{q}_2$

$$\underline{u}_0 = \underline{Q} \begin{bmatrix} Y_1(0) \\ Y_2(0) \end{bmatrix}, \quad \dot{\underline{u}}_0 = \underline{Q} \begin{bmatrix} \dot{Y}_1(0) \\ \dot{Y}_2(0) \end{bmatrix}$$

$Y_1(t) = c_1 \cos t + c_2 \sin t; \quad \dot{Y}_1(t) = -c_1 \sin t + c_2 \cos t$

$Y_2(t) = d_1 \cos 3t + d_2 \sin 3t; \quad \dot{Y}_2(t) = -3d_1 \sin 3t + 3d_2 \cos 3t$

$Y_1(0) = c_1, \quad Y_2(0) = d_1, \quad \dot{Y}_1(0) = c_2, \quad \dot{Y}_2 = 3d_2$