Math 630-102
Lecture 4 • Victor Matveev
Fundamental spaces in the general $m \times n$ case

1. The null-space N(A), the colunn space C(A), and the row-space $C(A^T)$ are completely different animals:

A. **C(A)** is where the **matrix columns** and **attainable right-hand sides** live. Therefore, C(A) is a subspace in R^m (m = number of equations). The dimension of C(A) is the number of independent columns forming the basis; it is called the rank of the matrix:

   i) **dim C(A) = rank A = r**.
   ii) **rank A** $\le$ ***m*** since C(A) is a subspace of R^m
   iii) **rank A** $\le$ ***n*** since there are only n columns available

B. **N(A)** is where the **solutions** to $Ax_N=0$ live. So if you can solve this equation, you know the null-space – nothing to it. N(A) must be a subspace of R^n (n = number of unknowns). It's dimension is the number of independent solutions to $Ax_N=0$ (set each free variable in turn to one, the rest to zero), and it satisfies:

   iv) **dim N(A) =** $\boldsymbol{n - r}$ (the number of free variables). Therefore, dim C(A) + dim N(A) = n

C. $\mathbf{C(A^T)}$, the row-space, is where the **matrix rows** live. Therefore, it is a subspace of R^n, just like N(A) is, and its dimension is the number of independent rows forming the basis. Since the number of independent rows equals the number of independent columns, we have:

   v) $\mathbf{dim\ C(A^T) = dim\ C(A) = rank\ A} = \boldsymbol{r}$

   As the matrix-vector multiplication in equation $Ax_N = 0$ involves the product of rows with the null-space elements x_N, and equals zero, the row-space is perpendicular to the null-space. In particular, their bases are mutually orthogonal. As I will show, this means that the dimensions of $C(A^T)$ and N(A) must add up to n (think of a plane and its perpendicular line in R^3, both containing the origin – if one of them is N(A), the other must be $C(A^T)$):

$$\dim C(A^T) + \dim N(A) = r + \dim N(A) = n$$

   Note that this agrees with equation (iv) above.

2. Finding bases of C(A), N(A) and C(A^T), and the general solution to Ax_b = *b*

The technique itself is straight-forward: try to solve the system using row operations. To find the particular solution for non-zero *b*, perform operations on *b* as well (use augmented matrix for simplicity), as usual. For N(A), the right-hand side always remains zero, of course. *Don't forget the row exchanges, if needed!*

Step 1. Basis of C(A) and C(A^T)

Step one is similar to LU factorization, except that in general you obtain U in the **row-echelon form**. *Note where the pivots are!* You are done as far as C(A) and C(A^T) are concerned. The **C(A) basis is formed by the pivot-containing columns of the original matrix A** (not U – row operations change the column space!). The C(A^T) basis is formed by the pivot-containing rows of either U or A (doesn't matter, both work) – this is because row operations cannot change the row space (think about it!)

Step 2. Basis of N(A)

Step two is similar to back-substitution in the Gauss-Jordan method, except that on the left we end up with the **reduced row-echelon form matrix R**, not the identity matrix (if you end up with an identity matrix, that means that A is a non-singular square matrix, and the null-space is empty). Otherwise, the simplest-looking basis of N(A) is found by setting each free variable in turn to 1, the other free variables to zero, and solving for the pivot variables. Note that pivot variables will equal the negatives of the non-pivot columns coefficients. We are done with N(A)! We could set free variables to some other values, of course – the basis is never unique.

Step 3. General solution of Ax_b = *b*

Perform above steps with the column b in the augmented matrix. Once you obtain R, set the free (non-pivot) variables to zero (for simplicity), and read out the solution. You will see that your pivot variables simply equal the right-hand side column. This will give you a particular solution x_p. To find the most general solution, add the general null-space element:

$$x_b = x_p + x_N$$