Math 630-102

Midterm Exam

March 8, 2007

This is a closed-book exam: notes and calculators are *not* allowed. Please explain your solutions to receive full credit. Use only the methods learned so far (no determinants please). Check your answers.

1. (20) Use the Gauss-Jordan method to find the inverse of matrix A, and in the process determine the LU factorization for this matrix. Please state clearly the row operations.

   A= $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 7 & 5 \end{bmatrix}$. Finally, use the obtained inverse A^{-1} to solve A x = $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. (24) Consider the linear system Ax=b with

   $$A=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \quad b=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   a) Find the reduced row-echelon form (“R”) of the matrix A
   b) Are the rows of A linearly independent? Are the columns of A linearly independent?
   c) Find the dimensions and the bases of the four fundamental subspaces of A.
   d) What are the conditions on b_1, b_2 and b_3, if any, under which the solution exists?
   e) Find the general solution to A x = b for b_1=1, b_2 =4, and b_3=5 (to receive full credit, use only row operations for back-substitution). Check your answer!

3. (14) Find the basis of the orthogonal complement to the space spanned (generated) by the vector (1, 3, 2). Describe the geometry of the space spanned by this vector, and of its orthogonal complement: is it a point, a line, or a plane?

4. (12) Suppose T is a transformation that transforms an $\mathbf{R}^3$ vector x= (x_1, x_2, x_3) into an $\mathbf{R}^2$ vector y according to the rule

   $$x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \xrightarrow{T} y=\begin{bmatrix} x_1+x_2+x_3 \\ x_2+x_3 \end{bmatrix}$$

   a) Verify that this transformation satisfies the linearity properties

b) Find the effect of this transformation on the standard basis of $\mathbf{R}^3$, and use your result to construct the transformation matrix. Note that the transformation matrix is not square in this case.

5. (12) Use the Gram-Schmidt method to find the orthonormal basis for C(A) of matrix

$$A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Check your answer by calculating Q^TQ, where Q is the matrix whose columns are composed of the orthonormal basis vectors (note that Q is not a square matrix in this problem). Write an expression for a projection matrix that projects any $\mathbf{R}^3$ vector onto C(A) (hint: it will be a sum of two different projection matrices).

6. (9) If ABC=D, find the expression for B. Assume all four matrices are square and non-singular.

7. (9) If we have to exchange some rows of a general *m* by *n* matrix A in order to perform some row operations, which of the four fundamental subspaces of the matrix will ***not*** be affected by this row exchange? Please explain.