Math 630 Linear Algebra 1/2

Quiz # 5

Problem 1

$$\det(\underline{A} - \lambda \underline{I}) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0, \quad \lambda^2 - 5\lambda + 6 = 0$$

$\lambda_1 = 2$, $\lambda_2 = 3$.

$$\lambda_1 = 2 \Rightarrow (\underline{A} - 2\underline{I})\underline{x}_1 = \underline{0} \Rightarrow \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \underline{x}_1 = \underline{0}$$

$$\underline{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 3 \Rightarrow (\underline{A} - 3\underline{I})\underline{x}_2 = \underline{0} \Rightarrow \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} \underline{x}_2 = \underline{0}$$

$$\underline{x}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$\underline{B} = \underline{A} - 7\underline{I}$ has $\lambda_1 = 2-7 = -5$, $\lambda_2 = 3-7 = -4$.

The eigenvectors of $\underline{A}$ & $\underline{B}$ are the same.

of course $\lambda_1 + \lambda_2 = \text{trace}(\underline{A}) = 5$, $\lambda_1 \lambda_2 = \det(\underline{A}) = 6$

Problem 2

$$\underline{A} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \quad \lambda_1 = 0, \lambda_2 = 2. \quad \underline{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \underline{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underline{A} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}^{-1}$$

$$\underline{B} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -5 & 18 \\ -3 & 10 \end{pmatrix}$$

Problem 3

$$G_{k+2} = (G_{k+1} + G_k)/2, \quad Y_{k+1} = \begin{pmatrix} G_{k+2} \\ G_{k+1} \end{pmatrix}, \quad Y_k = \begin{pmatrix} G_{k+1} \\ G_k \end{pmatrix}$$

$$\underline{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \quad \underline{Y}_{k+1} = \underline{A}\,\underline{Y}_k$$

$$\lambda_1 = -\frac{1}{2}, \quad \underline{x}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \quad \lambda_2 = 1, \quad \underline{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{S} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}, \quad \underline{S}^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & -1 \end{pmatrix}\left(-\frac{1}{3}\right)$$ 2/2

$$\underline{\Lambda} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{Y}_k = \underline{S}\,\underline{\Lambda}^k \underline{S}^{-1} \underline{Y}_0 = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} (-\frac{1}{2})^k & 0 \\ 0 & 1^k \end{pmatrix}\begin{pmatrix} -1/3 & +1/3 \\ +2/3 & +1/3 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$k \to \infty \qquad \underline{Y}_k = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \qquad G_k = \frac{1}{3}$

Problem 4

$\underline{A}_1$ and $\underline{A}_2$ not diagonalizable

$\underline{A}_3$ diagonalizable.

$\underline{K} = \begin{pmatrix} i & i \\ i & i \end{pmatrix}, \qquad \lambda_1 = 0, \quad \lambda_2 = 2i$

$\underline{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \underline{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\underline{K} = \underline{S}\,\underline{\Lambda}\,\underline{S}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}/2$$

$$e^{\underline{K}t} = \underline{S}\, e^{\underline{\Lambda}t} \underline{S}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{2it} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} 1/2$$

$(e^{\underline{K}t})^H = e^{\underline{K}^H t} = e^{-\underline{K}t} \Rightarrow e^{\underline{K}t}$ unitary.

$$\frac{d}{dt}(\underline{e}^{\underline{K}t})\Big|_{t=0} = \underline{K}\,\underline{e}^{\underline{K}t}\Big|_{t=0} = \underline{K}$$

Note that $\quad \underline{\Lambda}^H = -\underline{\Lambda}$, $\underline{e}^{\Lambda t}$ is also unitary.