Math 630 - Linear Algebra and Its Applications

Instructor: Prof. X. Sheldon Wang

Final Exam

(Closed book)

Assigned: 6:00pm, May 5th, 2005
Due: 8:30pm, May 5th, 2005

Problem 1 (10 points)

1) Solve the nonsingular system

$$\begin{aligned} 3u + 3v &= b_1 \\ 3u + 10v + 7w &= b_2 \\ 7v + 8w &= b_3. \end{aligned}$$

2) Use the pivots of $\mathbf{A} - 2\mathbf{I}$ to decide whether $\mathbf{A}$ has an eigenvalue smaller than 2.

Problem 2 (10 points)

If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A} + \mathbf{B}$ are invertible square matrices, find a formula for the inverse of $\mathbf{A}^{-1} + \mathbf{B}^{-1}$. Hint: Use $\mathbf{A}^{-1}(\mathbf{A} + \mathbf{B})\mathbf{B}^{-1}$.

Problem 3 (10 points)

1) Find all solutions to

$$\mathbf{Ux} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}.$$

2) If $\mathbf{A}$ has the same four fundamental subspaces as $\mathbf{B}$, does $\mathbf{A} = \mathbf{B}$? Give an example.

Problem 4 (10 points)

What are the intersections of the following pairs of subspaces? The plane perpendicular to $(1, 1, 0)$ and the plane perpendicular to $(0, 1, 1)$ in $\mathcal{R}^3$.

Problem 5 (10 points)

How far is the line $x_1 - x_2 = 8$ from the origin and what point on it is the nearest? Use project matrix and other linear algebra concepts.

Problem 6 (10 points)

Suppose you do two row operations at once, going from

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ to } \begin{bmatrix} a-mc & b-md \\ c-la & d-lb \end{bmatrix}.$$

Express the determinant of the second matrix as a function of the determinant of the first matrix.

Problem 7 (10 points)

If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times m$ show that

$$det \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ -\mathbf{B} & \mathbf{I} \end{bmatrix} = det\mathbf{AB}. \text{ (Hint: right-multiply by } \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B} & \mathbf{I} \end{bmatrix}.)$$

Do an example with $m < n$ and an example with $m > n$. Why does the second example have $det\mathbf{AB} = 0$?

Problem 8 (10 points)

Solve $\dfrac{d\mathbf{u}}{dt} = \mathbf{Pu}$ when $\mathbf{P}$ is a projection:

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \mathbf{u} \text{ with } \mathbf{u}_o = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Show that the column space component of $\mathbf{u}_o$ increases exponentially while the nullspace component stays fixed.

Problem 9 (10 points)

Given

$$\mathbf{u}_{k+1} = \mathbf{A}\mathbf{u}_k = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix} \mathbf{u}_k \text{ with } \mathbf{u}_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

1) In the range of $0 < a, b < 1$, the given equation is a Markov process. Computer $\mathbf{u}_k = \mathbf{S}\mathbf{\Lambda}^k\mathbf{S}^{-1}\mathbf{u}_o$ for any a and b within $(0, 1)$.
2) What is the limit of u_k as $k \to \infty$, when $a \neq b$?

Problem 10 (10 points)

What are the eigenvalues λ and frequencies ω for

$$\frac{d^2\mathbf{u}}{dt^2} = \begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix} \mathbf{u}.$$

Write down the general solution.