Math 630 - Linear Algebra and Its Applications

Instructor: Prof. X. Sheldon Wang

Mid-Term

(Closed book)

Assigned: 6:00pm, Mar. 9, 2006
Due: 8:00pm, Mar. 9, 2006

Problem 1 (15 points)

Construct a matrix with $(1, 0, 1)$ and $(1, 2, 0)$ as a basis for its row space and its column space. What is the rank of such a matrix? Why can't this be a basis for the row space and nullspace?

Problem 2 (15 points)

Find $\mathbf{L}$ and $\mathbf{U}$ for the nonsymmetric matrix:

$$\mathbf{A} = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Problem 3 (15 points)

Suppose $\mathbf{a}_1 = [1, 1, 0]^T$, $\mathbf{a}_2 = [0, -1, 0]^T$, and $\mathbf{b} = [2, 1, 4]^T$. Find x_1 and x_2 such that $\|x_1\mathbf{a}_1 + x_2\mathbf{a}_2 - \mathbf{b}\|$ is minimized.

Problem 4 (15 points)

Under what condition on the columns of $\mathbf{A}$ is $\mathbf{A}^T\mathbf{A}$ invertible? Without carrying out the matrix multiplication, find out if $\mathbf{A}^T\mathbf{A}$ based on the following matrix $\mathbf{A}$ is invertible. Find bases for the four fundamental subspaces of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Problem 5 (15 points)

Find an orthonormal basis for the plane $x - y + z = 0$, and find the matrix $\mathbf{P}$ which projects onto the plane. What is the nullspace of $\mathbf{P}$?

Problem 6 (15 points)

Let $\mathbf{A} = [3 \quad 1 \quad -1]$ and let V be the nullspace of $\mathbf{A}$. (a) Find a basis for V and a basis for $V^{\perp}$. (b) Write down an orthonormal basis for $V^{\perp}$, and find the projection matrix $\mathbf{P}_1$ which projects vectors in $\mathcal{R}^3$ onto $V^{\perp}$. (c) Find the projection matrix $\mathbf{P}_2$ which projects vectors in $\mathcal{R}^3$ onto V.

Problem 7 (10 points)

Show that the modified Gram-Schmidt steps

$\mathbf{c}'' = \mathbf{c} - (\mathbf{q}_1^T \mathbf{c})\mathbf{q}_1$ and $\mathbf{c}' = \mathbf{c}'' - (\mathbf{q}_2^T \mathbf{c}'')\mathbf{q}_2$

produce the same vector $\mathbf{c}'$ as the original Gram-Schmidt steps. The modified steps are much more stable with respect to round-off errors, to subtract off the projections one at a time.