Math 630 Linear Algebra
Mid-Term

Problem 1

$$\underline{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \qquad \underline{C} = \underline{A}\,\underline{A}^T$$

rank : 2 $\qquad$ row space $\perp$ null space .

Problem 2

$$\underline{A} = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}$$

Problem 3

$$\underline{A} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \underline{A}^T\underline{A} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad (\underline{A}^T\underline{A})^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\underline{P} = \underline{A}(\underline{A}^T\underline{A})^{-1}\underline{A}^T = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \qquad \underline{p} = \underline{P}\,\underline{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{A}\,\underline{x} = \underline{p} \text{ (why ?!)} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Problem 4

All columns of $\underline{A}$ are linearly independent $\Rightarrow$ 2
$\underline{A}^T\underline{A}$ invertible

zero row! $\leftarrow$ (why?)

The given $\underline{A}$ obviously does not satisfy this!

$$[\underline{A} \vdots \underline{I}] = \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \leftarrow N(A^T)$$

pivot

$$\Rightarrow \left[\begin{array}{cccccccc} 1 & 0 & -2 & 0 & 1 & -1 & 0 & -1/4 \\ 0 & 1 & 1 & 0 & 0 & 1/2 & 0 & -1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/4 \end{array}\right]$$

$\uparrow$ free variable

rank 3 $\quad \dim[N(\underline{A})] = 1$, $\quad \dim[N(\underline{A}^T)] = 1$ 2

$C(\underline{A})$: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 4 \end{bmatrix}$

$N(\underline{A})$: $\begin{bmatrix} -F \\ I_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ 2

$N(\underline{A}^T)$: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ 2

$Row(\underline{A})$: $[1\ 0\ -2\ 0]$, $[0\ 1\ 1\ 0]$, $[0\ 0\ 0\ 1]$ 2

blem 5

two Nullspace base vectors:

$$[1 \ -1 \ 1] \Rightarrow F = [-1 \ 1] \Rightarrow \underline{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \underline{a}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$\underline{q}_1 = \frac{\underline{a}_1}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \hat{\underline{a}}_2 = \underline{a}_2 - \underline{q}_1^T \underline{a}_2 \underline{q}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} (-1/\sqrt{2})$$

3

$$= \begin{bmatrix} -1/2 \\ +1/2 \\ 1 \end{bmatrix} \Rightarrow \underline{q}_2 = \frac{\hat{\underline{a}}_2}{\|\hat{\underline{a}}_2\|} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 2/\sqrt{3} \end{bmatrix} / \sqrt{2}$$

$$\underline{A} = \begin{bmatrix} \underline{q}_1 & \underline{q}_2 \end{bmatrix}, \quad \underline{P} = \underline{A}(\underline{A}^T\underline{A})^{-1}\underline{A}^T = \underline{Q}\underline{Q}^T = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

3

$N(\underline{P})$ is the row space! $[1 \ -1 \ 1]$ 3

Problem 6

(a) a basis for V: 3

$[3 \ 1 \ -1]$ $\qquad F = [3 \ -1]$, $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

↑ pivot.

a basis for $V^\perp$. $[3 \ 1 \ -1]$

(b) $\underline{q} = \begin{bmatrix} \frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ -\frac{1}{\sqrt{11}} \end{bmatrix}, \quad \underline{P}_1 = \underline{q}\,\underline{q}^T = \frac{1}{11}\begin{bmatrix} 9 & 3 & -3 \\ 3 & 1 & -1 \\ -3 & -1 & 1 \end{bmatrix}$ 3

(c) $\underline{P}_2 = \underline{I} - \underline{P}_1 = \frac{1}{11}\begin{bmatrix} +2 & -3 & +3 \\ -3 & +10 & +1 \\ +3 & +1 & -1 \end{bmatrix}$

or. based on $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \underline{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \\ 0 \end{bmatrix}, \quad \underline{q}_2 = \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - [0 \ 1 \ 1]\begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \\ 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \\ 0 \end{bmatrix}}{\| \quad \|} = \frac{1}{\sqrt{11}}\frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \\ 10 \end{bmatrix}$

$\underline{P}_2 = \underline{Q}\,\underline{Q}^T$ with $\underline{Q} = [\underline{q}_1 \ \underline{q}_2]$, same results!

Problem 7

$$\ast \begin{cases} \underline{c}'' = \underline{c} - (\underline{q}_1^T \underline{c})\,\underline{q}_1 \\ \underline{c}' = \underline{c}'' - (\underline{q}_2^T \underline{c}'')\,\underline{q}_2 \end{cases}$$

modified GS

$$\ast \quad \hat{\underline{c}} = \underline{c} - (\underline{q}_1^T \underline{c})\,\underline{q}_1 - (\underline{q}_2^T \underline{c})\,\underline{q}_2$$

GS

Note $\quad \underline{q}_2^T \underline{q}_1 = 0$

round-off $\leftarrow \underbrace{10^{-8} \quad 10^{8}}^{\text{trouble!}}$

$$\underline{c}' = \underbrace{\underline{c} - (\underline{q}_1^T \underline{c})\,\underline{q}_1 - (\underline{q}_2^T \underline{c})\,\underline{q}_2}_{\text{is same as } \hat{\underline{c}}} + (\underline{q}_2^T \underline{q}_1)\,\underline{q}_1^T \underline{c}$$

large components in $\underline{q}_1$

Theoretically zero, numerically no

(not needed for exam)