Numerical study of periodic instanton configurations in two-dimensional abelian Higgs theory

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Numerical minimization of the euclidean action of the two-dimensional abelian Higgs model is used to construct periodic instantons, the euclidean field configurations with two turning points describing transitions between the vicinities of topologically distinct vacua. Periodic instantons are found at any energy (up to the sphaleron energy E_{sph}) and for a wide range of parameters of the theory. We obtain the dependence of the action and the energy of the periodic instanton on its period; these quantities directly determine the probability of certain multiparticle scattering events.

1. In the four-dimensional standard electroweak theory, instanton-induced transitions account for interesting phenomenon of baryon- and lepton-number violation; here we consider a much simpler two-dimensional abelian Higgs theory which is analogous to the former to the extent that it also possesses a non-trivial vacuum structure. In both theories, one may anticipate the existence of periodic instanton configurations, which are solutions to the euclidean field equations with two turning points and zero winding number [1] (see also [2,3]). As is shown by Khlebnikov et al. [1], the periodic instanton is an exact saddle point in the functional integral for the probability of the multiparticle scattering event that leads to a transition between the vicinities of topologically distinct vacua. It is also shown that the analytic continuation of the periodic instanton to the Minkowski domain through its turning points provides the most probable initial and final states of this transition. In other words, the transition induced by the periodic instanton has the largest probability at a given energy. In the semiclassical approximation, this probability was found to be [1]

$$\sigma_E = \exp\{-S + ET\}, \tag{1}$$

where T is the period and S is the action per period of the periodic instanton of energy E.

At low energies, the periodic instanton configuration can be approximated by a temporal alternating sequence of instantons and anti-instantons separated by equal intervals of $T/2$. The period T grows exponentially as the energy approaches zero.

At energies close to the sphaleron [4] energy E_{sph}, the periodic instanton can be described as the sum of the static sphaleron configuration and the oscillation in the negative eigenmode around the sphaleron [5,6]. As the energy E approaches E_{sph}, the amplitude of this oscillation vanishes. The period of the periodic instanton in this approximation is equal to the period of this eigenmode.

At arbitrary energies, the analytic form of the periodic instanton is not known; furthermore, the boundary conditions (fields at the turning points) are also unknown. In the present paper we evaluate numerically the periodic instantons of the two-dimensional abelian Higgs theory at any energy and for wide range of parameters of the theory. We confirm the expected properties of these configurations and show, in particular, that the semiclassical probability σ_E, eq. (1), interpolates between its instanton value and unity as the energy varies from zero to E_{sph}.

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2. The lagrangian of the two-dimensional abelian Higgs model reads

$$L = -\tfrac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu - igA_\mu)\phi|^2 - \lambda(|\phi|^2 - \tfrac{1}{2}v^2)^2,$$

$$\mu,\nu = 0,1.$$

Here ϕ is the complex Higgs field and A_μ is the U(1) gauge field. The particle spectrum of this model consists of the Higgs boson of mass $M_H^2 = 2\lambda v^2$ and the vector boson of mass $M_W^2 = g^2 v^2$. Let us consider the temporal gauge $A_0 = 0$; the remaining symmetry we will fix according to the condition

$$A_1(x, t = t_0) = 0.$$

Later we will define the value of t_0.

In this gauge we may choose the following set of topologically distinct vacua

$$\phi_n(x) = v\exp[i\alpha_n(x)], \quad A_1^n(x) = \frac{1}{g}\partial_1\alpha_n,$$

$$(\alpha_n(x = +\infty) - \alpha_n(x = -\infty)) = 2\pi(n + \tfrac{1}{2}),$$

$$\phi_n(-\infty) = -v, \quad \phi_n(+\infty) = v.$$

This set is related to a conventional one with $\alpha_n(x = +\infty) - \alpha_n(x = -\infty) = 2\pi n$ by a time-independent gauge transformation that plays no role in any physical process.

In the space of field configurations, the neighboring vacua are separated by a static energy barrier. The top of this barrier is associated with the sphaleron [4], which is the static unstable solution of the field equations. In our gauge this solution is the ordinary kink (cf. [5,7])

$$A_\mu^{\text{sph}} = 0, \quad \phi^{\text{sph}} = \frac{v}{\sqrt{2}}\tanh\frac{M_H x}{2}.$$

Its energy (the height of the barrier) is equal to

$$E_{\text{sph}} = \tfrac{1}{3}\sqrt{8\lambda}\, v^3.$$

Among the modes of oscillations of fields around the sphaleron there is one negative eigenmode, which makes the sphaleron unstable. The frequency of this mode is [6]

$$\omega_-^2 = -\frac{M_H^2}{8}\left(\sqrt{1 + \frac{16M_W^2}{M_H^2}} + 1\right). \tag{2}$$

This quantity determines the minimal period of a periodic instanton configuration: as energy increases from zero to E_{sph} the period changes from infinity to $T_- = 2\pi/|\omega_-|$.

The instanton solution of this model is the well-known Abrikosov–Nielsen–Olesen vortex [8,9]. The instanton is a zero-energy, finite action euclidean solution to the field equations interpolating between two neighboring vacua. Its winding number is equal to 1,

$$\frac{1}{2\pi}\oint A_\mu^I\,dx_\mu = \frac{1}{2\pi}\int A_1^I\,dx^1\Big|_{t=+\infty}^{t=-\infty} = 1.$$

At various values of M_H/M_W this solution was obtained numerically in ref. [10].

3. We construct the periodic instanton field for various values of the period (ranging from T_- to infinity) using the numerical minimization of the euclidean action. Thus, the values of energy and action of the periodic instanton and its other properties are obtained as functions of its period.

The minimization is performed on the space of fields satisfying the following conditions:

$$\dot{A}_1(x, -\tfrac{1}{2}T) = \dot{A}_1(x, 0) = 0,$$

$$A_1(-\infty, t) = A_1(+\infty, t) = 0, \tag{3}$$

$$\dot{\phi}(x, -\tfrac{1}{2}T) = \dot{\phi}(x, 0) = 0,$$

$$\phi(-\infty, t) = -v, \quad \phi(+\infty, t) = v, \tag{4}$$

$$A_1(x, t_0 = -\tfrac{1}{4}) = 0. \tag{5}$$

Notice that Gauss' law $\partial_1\partial_0 A_1 = -\tfrac{1}{2}ig(\dot{\phi}\phi^* - \phi\dot{\phi}^*)$ is satisfied at the turning points; this provides that it will be automatically satisfied at all times for the fields minimizing the action and obeying the field equations.

Eq. (5) specifies our choice of gauge. This particular choice leads to the following additional symmetries of the solution:

$$A_1(x, -\tfrac{1}{4}T - t) = -A_1(x, -\tfrac{1}{4}T + t),$$

$$\phi(x, -\tfrac{1}{4}T - t) = \phi^*(x, -\tfrac{1}{4}T + t).$$

This symmetry allows one to perform minimization on the time interval equal to the quarter of the period, which is convenient for the numerical study.

The calculations are performed on a two-dimensional lattice with the time dimension of $[-\frac{1}{4}T, -\frac{1}{2}T]$; the choice of the spatial x-dimension depends on the desired accuracy and the values of parameters M_H and M_W.

First, an arbitrary configuration satisfying the conditions (3)–(5) is chosen. Then the following step is repeated as often as necessary:

The trial function to be added to one of the fields is chosen as the product of a random time-dependent trigonometric harmonic and x-dependent gaussian function of random width and position. Then a standard single-parameter minimization technique is applied with respect to the amplitude of this trial function. Along with the trial function its derivatives of a known analytical form are also calculated and added after minimization to the corresponding derivatives of the field. In this sense the derivatives are calculated "exactly": we do not use any difference approximations. This greatly increases the accuracy of the results.

4. Calculations were performed for different values of the ratio M_H/M_W ranging from $\frac{1}{4}$ to 4. Here we present the results mainly for the case $M_H = M_W = 1$; the results at other values of M_H/M_W are similar.

Figs. 1 and 2 show respectively the dependence of the energy and the action per period of the periodic instanton on its period. In full agreement with the expectations, the energy approaches E_{sph} as the period decreases. If the period is set to a value smaller then T_-, calculations lead to a static sphaleron con-
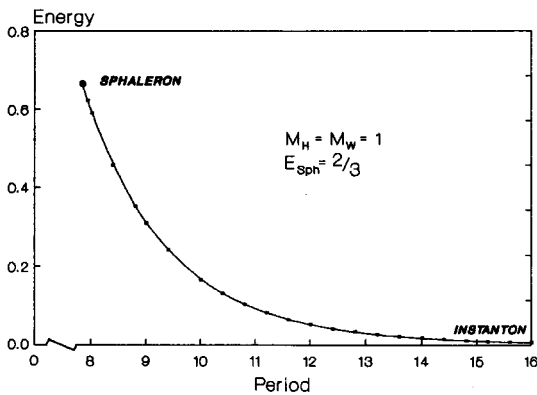

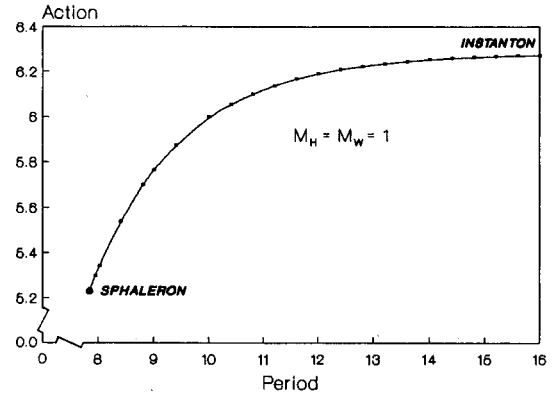Fig. 1. Dependence of the energy of the periodic instanton on its period.

Fig. 2. Dependence of the action per period of the periodic instanton on its period.
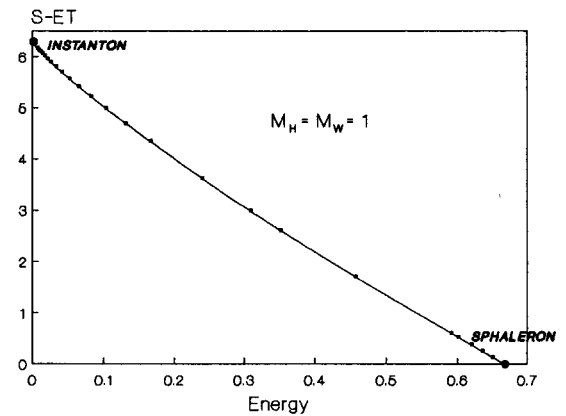

Fig. 3. Dependence of the value $(S - ET)$ on the energy.

figuration, since there are no other periodic solutions at $T < T_-$. The numerically obtained minimal value of the period, T_-, coincides with the expected value, determined by eq. (2). At large values of the period the energy is close to zero, and the action is close to twice the value of the action of the instanton (which is equal to π when $M_H = M_W = 1$). The functions $S(T)$ and $E(T)$ satisfy the following relation [1]:

$$E(T) = \frac{\partial S(T)}{\partial T}, \qquad (6)$$

which provides an additional check for numerical study.

Fig. 3 shows the behavior of $(S - ET) = -\ln \sigma_E$ as the function of energy, where σ_E is the probability of the instanton-induced multiparticle scattering event, see eq. (1). At $E = E_{\text{sph}}$ the value of $(S-ET)$ reaches
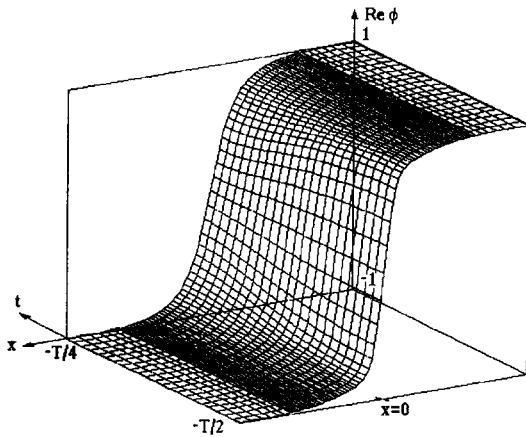
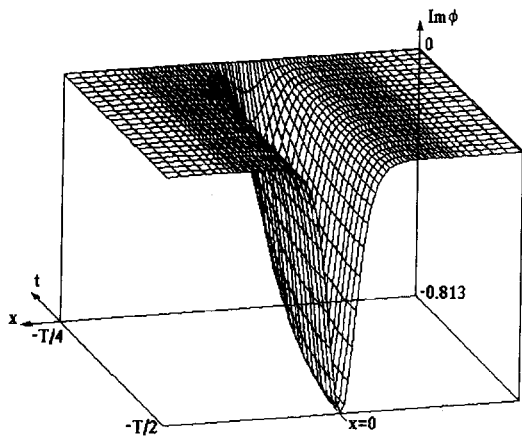Fig. 4. Real part of the Higgs field for $M_W = 1$, $M_H = \frac{1}{4}$, $T = 32$.

Fig. 5. Imaginary part of the Higgs field for $M_W = 1$, $M_H = \frac{1}{4}$, $T = 32$.

zero, as should have been expected; this corresponds to an unsuppressed probability of the transition.

For completeness, in figs. 4–6 we show typical periodic instanton field configurations. Here the fields are shown for the case $M_W = 1$, $M_H = \frac{1}{4}$. We have checked that at large T these configurations do indeed coincide with the instanton–anti-instanton pair.

5. To conclude, we have verified numerically the existence of periodic instantons in a two-dimensional abelian Higgs model. They indeed interpolate between a widely separated chain of instantons and anti-instantons and the sphaleron as energy increases from
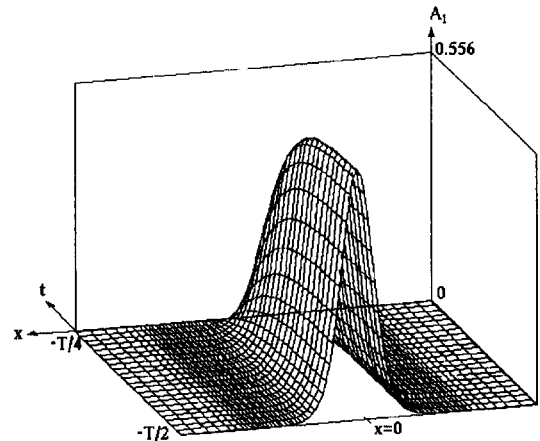

Fig. 6. Spatial component of the gauge field $A_1(x, t)$ for $M_W = 1$, $M_H = \frac{1}{4}$, $T = 32$.

zero to E_{sph} and describe most favorable tunneling events at each energy. The probability of such event increases with energy starting from $\exp\{-2S_{\text{inst}}\}$ at $E = 0$ and becomes no longer exponentially suppressed at $E = E_{\text{sph}}$.

References

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