2. $\mathcal{X} = \{0, \ldots, 23\} = \mathcal{Y}$

$\uparrow$ 12am $\quad \uparrow$ 11pm

$Y = X + Z \mod 24 \quad$ with $\quad Z \sim p(z) = \begin{cases} 1/2 & z = 0 \\ 1/4 & z = \pm 1 \end{cases}$

This is a symmetric channel and

$$C = \log_2(24) - H\left(\left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}\right)\right)$$

$$= \log_2(24) - \frac{3}{2} = 3.085$$

1. $A = \begin{cases} 1 & \text{wp } 1/3 \\ 2 & \text{wp } 2/3 \end{cases}$

$X_{i1} \sim$ Markov chain $(1/3, 1/3)$

$X_{i2} \underset{\text{iiD}}{\sim} \text{Ber}(1/2)$

independent

$Y_i = X_{iA}$ (stationary process)

a. $$H(\mathcal{Y}) = \lim_{n\to\infty} \frac{H(Y^n)}{n} = \lim_{n\to\infty} \frac{H(Y^n|A)}{n} + \frac{I(A;Y^n)}{n}$$

$\frac{I(A;Y^n)}{n} \to 0$ for $n \to \infty$ since $I(A;Y^n) \le H(A)$

$$= \frac{1}{3} H\left(\frac{1}{3}\right) + \frac{2}{3}$$

b. Shannon coding on long blocks achieves the entropy rate for any stationary process.

3. No, since $p(x^4, y^4) = \frac{1}{24}\, p(y_1|x_1)\, p(y_2|x_2)\, \underbrace{p(y_3|x_3)}_{=0}\, p(y_4|x_4)$

and hence $-\frac{1}{4} \log p(x^4, y^4) = \infty$

4. This follows as:

$$V^m - X^n - Y^n - \hat{V}^m \quad \text{Markov chain}$$

$$\Rightarrow \quad I(V^m;\hat{V}^m) \le I(V^m;Y^n)$$

But $\quad I(V^m;Y^n) \le \sum_{i=1}^{n} H(Y_i) - H(Y_i \mid V^m, X_i, Y^{i-1})$

Cond. reduces entropy

$$= \sum_{i=1}^{n} I(X_i;Y_i) \le nC$$

memoryless channel

and $\quad I(V^m;\hat{V}^m) = H(V^m) - H(V^m \mid \hat{V}^m)$

$$\ge mH(V) - m\delta(\epsilon)$$

Fano

$$\Rightarrow \quad mH(V) \le nC$$

5. By the AEP, we know that

$$\Pr\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i^2 - 1\right| \leq \epsilon\right] \to 1 \quad \text{as } n \to \infty$$

Therefore,

$$\Pr\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 < 1.1\right] \to 1 \quad \text{as } n \to \infty$$

and $\Pr\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 < 0.9\right] \to 0$ as $n \to \infty$

6. $p(x) = e^{\lambda_0 + \lambda x^2}$ $\quad x = 0, 1, 2$

$$\sum_{x=0}^{2} p(x) = e^{\lambda_0} + e^{\lambda_0} \cdot e^{\lambda} + e^{\lambda_0} \cdot e^{4\lambda} = 1 \Rightarrow e^{\lambda_0} = \frac{1}{1 + e^{\lambda} + e^{4\lambda}}$$

$$\sum_{x=0}^{2} x^2 p(x) = e^{\lambda_0} e^{\lambda} + 4 e^{\lambda_0} e^{4\lambda} = 1$$

$$\Rightarrow \frac{e^{\lambda} + 4 e^{4\lambda}}{1 + e^{\lambda} + e^{4\lambda}} = 1$$

$$\Rightarrow \cancel{e^{\lambda}} + 4 e^{4\lambda} = 1 + \cancel{e^{\lambda}} + e^{4\lambda} \Rightarrow e^{4\lambda} = \frac{1}{3}$$

$$\Rightarrow e^{\lambda_0} = \frac{1}{1 + \left(\frac{1}{3}\right)^{1/4} + \frac{1}{3}} = 0.4777$$

$$\Rightarrow p(x) = \begin{cases} 0.4777 & \text{for } x = 0 \\ 0.3630 & \text{for } x = 1 \\ 0.1592 & \text{for } x = 2 \end{cases}$$

7. $$R(D) = \min_{\substack{p(\hat{x}|x):\\ E[d(X,\hat{X})] \le D}} I(X;\hat{X})$$

$$= \min h(X) - h(X|\hat{X})$$

$$\le h(X) - \frac{1}{2}\log_2(2\pi e D)$$

$$h(X|\hat{X}) \le h(X-\hat{X}) \le \frac{1}{2}\log_2(2\pi e D)$$

$$\Rightarrow D \ge \underbrace{\frac{2^{2h(X)}}{2\pi e}}_{\text{entropy power}} 2^{-2R}$$

8. $$d(x,\hat{x}) = \begin{array}{c} 0 \\ ? \\ 1 \\ \\ \end{array}\!\!\begin{array}{c} \begin{bmatrix} 0 & \infty \\ 0 & 0 \\ \infty & 0 \end{bmatrix} \\ \begin{array}{cc} 0 & 1 \end{array} \end{array}$$

$$p(\hat{x}|x) = \begin{array}{c} 0 \\ ? \\ 1 \\ \\ \end{array}\!\!\begin{array}{c} \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} \\ \begin{array}{cc} 0 & 1 \end{array} \end{array}$$ ← same value by symmetry

$$D = E[d(X,\hat{X})] = 0$$

$$\begin{aligned} R = I(X;\hat{X}) &= H(X) - H(X|\hat{X}) \\ &= H(X, 1(X=?)) - H(X|\hat{X}) \\ &= H(1(X=?)) + H(X|1(X=?)) - H(X|\hat{X}) \\ &= H(q) + (1-q) - H(X|\hat{X}) \\ &= 1-q \end{aligned}$$

$$p(x|\hat{x}=0) = \begin{cases} \frac{1-q}{2} \cdot 1 / \frac{1}{2} = 1-q & x=0 \\ q\,\frac{1}{2} / \frac{1}{2} = q & x=? \end{cases}$$

$\Rightarrow R(D) = 1-q$ for $D \geq 0$

Compared with the entropy $H(X) = H(q) + (1-q)$, we see that under the given distortion metric, the decoder does not have to be informed about the position of the "?" symbols.