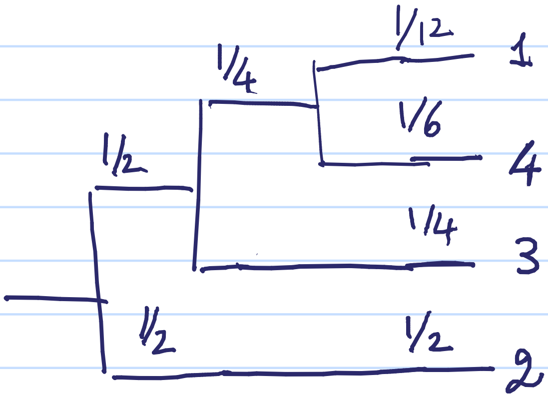
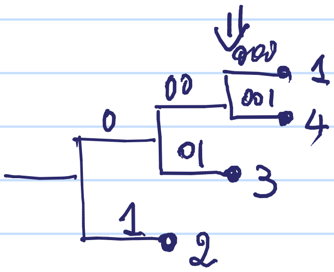
3 $p(x) = (1/12, 1/2, 1/4, 1/6)$

$q(x) = (0, 1/2, 1/4, 1/4)$

a. $q(x)$ correct pmf, $p(x)$ assumed





x	$c(x)$
1	000
2	1
3	01
4	001

$$L(C) = \frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{4} = \frac{7}{4}$$

$$H(X) = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2\cdot 2 = \frac{3}{2} \Rightarrow L(C) - H(X) = \frac{1}{4}$$

b. $D(q\|p) = \frac{1}{4}\log_2\frac{1/4}{1/6} = \frac{1}{4}\log_2\frac{3}{2} = 0.1462$

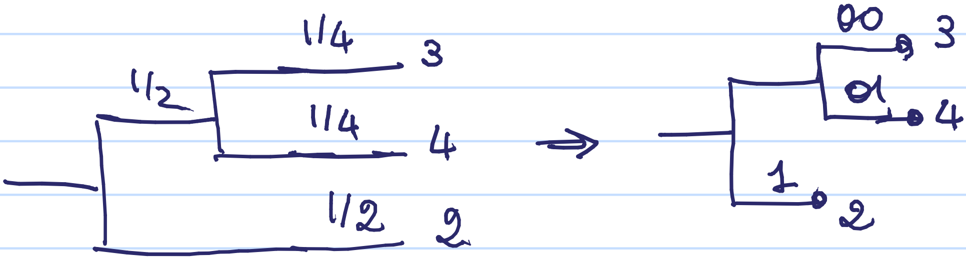
Shannon code: $L(C) = 0 + \frac{1}{2}\cdot 1 + \frac{1}{4}2 + \frac{1}{4}4 = 2$

$$L(C) - H(X) = \frac{3}{4}$$

We know that for Shannon codes:

$$0.1462 = D(q\|p) \leq \underbrace{L(C) - H(X)}_{3/4 = 0.75} < D(q\|p) + 1 = 1.1462$$

c. $p(x)$ correct pmf, $q(x)$ assumed



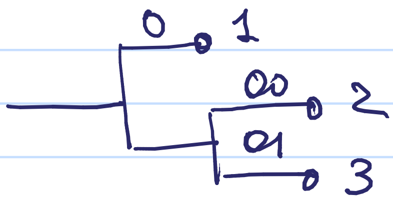
x	$c(x)$
1	no codeword
2	1
3	00
4	01

The code fails to represent "1" symbols.

$D(q \| p) = \infty$

4. $Y \sim p(x) = (1/2, 1/4, 1/4)$

Huffman code :



Use the following procedure

- Toss first coin X_1
- If $X_1 = 0$, set $Y = 1$
- Else toss second coin X_2.
  If $X_2 = 0$ set $Y = 2$
  else set $Y = 3$

It is easy to see that Y has the desired pmf. Moreover,

$$E[\#\text{ coin tosses}] = 1/2 \cdot 1 + \frac{1}{4} \cdot 2 \cdot 2 = \frac{3}{2} \; (= H(Y))$$

5. Show that $\ell(x) = \lceil -\log_2 p(x) \rceil + 1$, $c(x) = \lfloor \bar{F}(x) \rfloor_{\ell(x)}$ guarantee the prefix-free condition in arithmetic coding.

Please see notes.

2. $H(X_i \mid X^{i-1}) \leq H(X_i \mid X_2^{i-1}) = H(X_{i-1} \mid X^{i-2})$

conditioning reduces entropy

stationarity

1. $X \sim Bn(0.1)$

$Y = X + Z$, $Z \sim Bn(0.3) \Rightarrow \mathcal{Y} = \{0, 1, 2\}$

and $p(y|x=0) = (0.7, 0.3, 0)$

$p(y|x=1) = (0, 0.7, 0.3)$

Joint pmf $p(x,y)$:

X \ Y	0	1	2
0	0.63	0.27	0
1	0	0.07	0.03

a. $\hat{X} = \begin{cases} 0 & \text{if } Y=0 \text{ or } Y=1 \\ 1 & \text{if } Y=2 \end{cases}$

$P_e = \Pr[X \neq \hat{X}] = \Pr[X=1, Y=1] = 0.07$

b. $H(X|\hat{X}) = \underbrace{\Pr[\hat{X}=0]}_{0.97} H(X|\hat{X}=0) + \Pr[\hat{X}=1] \underbrace{H(X|\hat{X}=1)}_{=0}$

$$p(x|\hat{x}=0) = \frac{p(x, y=0) + p(x, y=1)}{p(y=0) + p(y=1)}$$

$$\Rightarrow p(x=1|\hat{x}=0) = \frac{0 + 0.07}{0.97} = 0.072$$

$$\Rightarrow H(X|\hat{X}) = 0.97 \times H(0.072) = 0.362$$

Fano: $\underbrace{H(X|\hat{X})}_{0.362} \leq \underbrace{H(P_e)}_{H(0.07) = 0.366}$ ✓

6. a. $-\frac{1}{k}\log_2 p(X^k) \xrightarrow[n\to\infty]{P} H(X)$ by the AEP

b. $\mathrm{var}\left(\frac{\ell(X^k)}{k}\right) = \mathrm{var}\left(-\frac{1}{k}\sum_{i=1}^{k}\log p(X_i)\right)$

$$= \frac{\mathrm{var}(-\log_2 p(X))}{k}$$

$$\mathrm{var}(-\log_2 p(X)) = E\left[(-\log_2 p(X))^2\right] - E\left[-\log_2 p(X)\right]^2$$

$$= p(\log_2 p)^2 + (1-p)(\log_2(1-p))^2 - H(p)^2$$

$$= 0.587$$

$$\Rightarrow \frac{0.587}{k} < 0.1 \Rightarrow k > 5.87 \Rightarrow k \geq 6$$