1.

a. $R \le H\left(\frac{1}{2}\right) + \frac{\log n + 3}{n} \xrightarrow[n\to\infty]{} H\left(\frac{1}{2}\right) = 1$ bit/source symbol

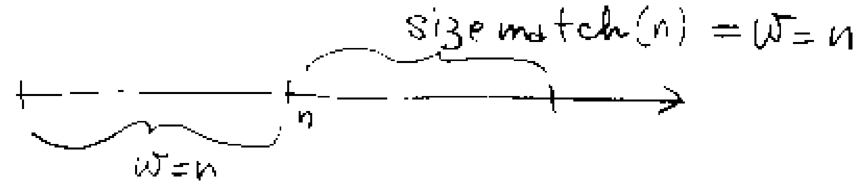
see notes

b. The scheme is universal for the class of memoryless sources. As such, it does not leverage the memory present in X^n.

c. size window

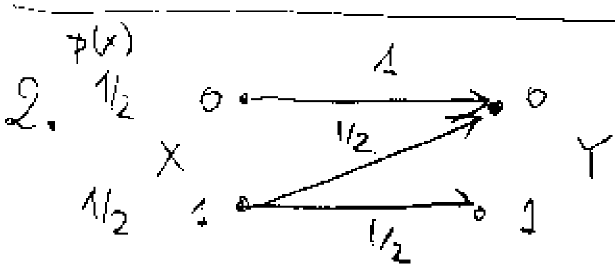
$$R = \lim_{n\to\infty} \frac{\log n + 2\log\log n}{\text{size match}(n)} = \lim_{n\to\infty} \frac{\log n + 2\log\log n}{n} = 0$$

see notes



d. LZ77 leverages the memory in the source.

2.



p(y)

$1/2 + \frac{1}{4} = \frac{3}{4}$

$\frac{1}{4}$

Maximum rate $= I(X;Y) = H(Y) - H(Y|X)$

$= H\left(\frac{1}{4}\right) - \frac{1}{2} H(Y|X=1)$

$= H\left(\frac{1}{4}\right) - \frac{1}{2} = 0.311$ bits/cu

3.

a. $\mathcal{C} = \{ \underbrace{(0,0,0)}^{x^3(1)}, \underbrace{(1,1,1)}^{x^3(2)} \}$ minimizes the prob. of error because the two codewords are at the maximum Hamming distance $(=3)$

b. $\Pr[\text{error}] = \Pr[\#\text{ bit flips} > 1] = 1 - \Pr[\#\text{bit flips} \leq 1]$

$$= 1 - (0.2 \times 0.8^2 \times 3 + 0.8^3)$$

$$= 0.104$$

The above probability follows by considering the optimal decoder that maps y^n to the closest received codeword in $\mathcal{C}$.

c. With 4 codewords, the Hamming dist between codewords < 3. The following codebook has Hamming distance 2 and is hence optimal

$$\mathcal{C} = \{ \overbrace{(0,0,0)}^{x^3(1)}, \overbrace{(0,1,1)}^{x^3(2)}, \overbrace{(1,0,1)}^{x^3(3)}, \overbrace{(1,1,0)}^{x^3(4)} \}$$

d. Optimal decoder:

y^3	decoded message $\hat{M}$
000	1
001	1 (arbitrarily assigned)
010	2 (arbitrarily assigned)
100	4 (arbitrarily assigned)
011	2
101	3
110	4
111	3 (arbitrarily assigned)

$\Pr[\text{Error}]$

$= \Pr[\text{Error} \mid M=1]$

$= 1 - (\Pr[\text{no flips}] + \frac{1}{3}\Pr[1 \text{ flip}])$

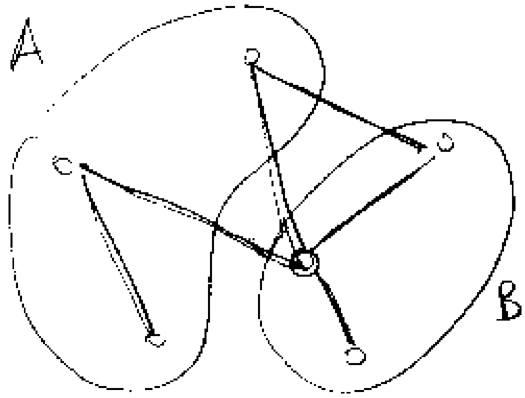
$= 1 - (0.8^3 + 0.8^2 .02)$

$= 0.36$

4. $\Pr[\text{Error}] = \Pr[3 \text{ erasures}] = \epsilon^3 = 0.2^3 = 0.008$

Smaller than for BSC.

5.



m edges
($ex: m = 6$)

$C(A,B)$ = number of edges connecting vertices in A and B
($ex: C(A,B) = 3$)

We want to show that it is always possible to partition the vertices so that $C(A,B) \geq \frac{m}{2}$.

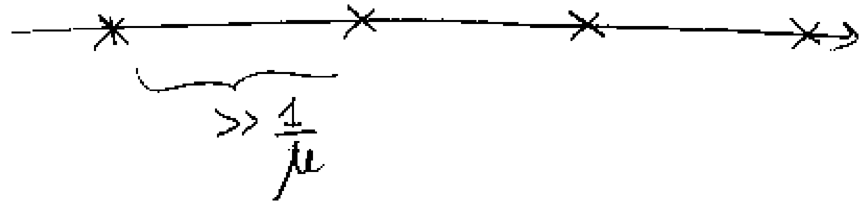
To this end, we use the probabilistic method.

- assign each vertex to A or B with equal probability and independently
- define
$$X_i = \begin{cases} 1 & \text{if edge } i \text{ connects a vertex in A and one in B} \\ 0 & \text{otherwise} \end{cases}$$
- Note that $C(A,B) = \sum_{i=1}^{m} X_i$ and
$$E[X_i] = \Pr[X_i = 1]$$
$$= \Pr[A \overset{i}{\text{—}} B \text{ or } B \overset{i}{\text{—}} A]$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
- So, we have
$$E[c(A,B)] = \sum_{i=1}^{m} E[X_i] = \frac{m}{2}$$

$\Rightarrow$ Since the average of $c(A,B)$ is $= m/2$ there must be at least one partition such that $C(A,B) \geq \frac{m}{2}$

6. a. $C = \infty$

It is enough to choose X such that



b. With cost constraint

$$I(X;Y) = h(Y) - h(Y|X) = h(Y) - h(Z)$$

$$= h(Y) - \log_2(e\mu)$$

$$\le \log_2(e(\lambda+\mu)) - \log_2(e\mu) = \log_2\left(1 + \frac{\lambda}{\mu}\right)$$

Max entropy lemma

$E[Y] = E[X] + E[Z]$

$= \lambda + \mu$

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7. $$I(X;Y,H) = \underbrace{I(X;H)}_{=0} + I(X;Y|H)$$

$$= I(X;Y) + \underbrace{I(X;H|Y)}_{\ge 0}$$

$$\Rightarrow I(X;Y|H) \ge I(X;Y)$$

8. We have

$$\underbrace{H\left(\tfrac{1}{3}\right)}_{\text{rate required by source}} \leq \underbrace{b}_{\text{bandwidth ratio}} \underbrace{(1-0.9)}_{\text{capacity of BEC}}$$

$$\Rightarrow \quad b \geqslant \frac{H\left(\frac{1}{3}\right)}{0.1} = 10 \times 0.918 = 9.18$$