ECE 776 - Final Spring 2014

1. (3 points) Two channels $p(y_1|x_1)$ and $p(y_2|x_2)$ can be used to communicate to the decoder but *not* at the same time. Specifically, the input to the channel is $X = (M, S)$, where $M \in \{1, 2\}$ selects which one of the two channels is used, and the output is distributed as $Y \sim p(y_1|s)$ if $M = 1$ and $Y \sim p(y_2|s)$ if $M = 2$.

a. Under the assumption that the output alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of the channels have an empty intersection, calculate the capacity (Hint: Your solution should depend on the capacities of the two channels, namely C_1 and C_2).

b. Check what happens when $C_1 = 0$ or $C_2 = 0$. Explain the result.

2. (2 points) Consider a BEC with erasure probability $p = 0.1$ and assume that the random codebook is generated with the capacity-achieving distribution $p(x)$.

a. Characterize the set of individually typical input and output sequences x^n and y^n as a function of n and ϵ.

b. Define as E^n the random process with $E_i = 1\{Y_i = e\}$. Relate the condition for x^n and y^n to be jointly typical with typicality conditions on E^n.

3. (2 points) A continuous random variable has pdf $f(x) = 1/2$ for $x \in [0, 1]$ and $x \in [2, 3]$ and $f(x) = 0$ elsewhere.

a. Sketch the typical set for $n = 2$.

b. Evaluate the approximate volume of the typical set obtained from the AEP for any n and compare it with the geometrical interpretation at point a.

4. (2 points) The Gaussian channel $Y = X + Z$ with $Z \sim \mathcal{N}(0, N)$ is given. No power constraint on X is imposed, but the constraint $E[Y^2] \leq Q$ is instead enforced. Calculate the capacity $C = \max_{p(x):E[Y^2]\leq Q} I(X; Y)$ as a function of Q.

5. (1 point) Find the maximum entropy distribution $p(x, y)$ with $\mathcal{X} = \{0, 1, 2\}$ and $\mathcal{Y} = \{0, 1, 2\}$, under the constraints the marginals are $p(x) = (1/2, 1/4, 1/4)$ and $p(y) = (1/3, 1/6, 1/2)$.

6. (1 point) Consider the maximum differential entropy pdf with support $\mathbb{R}$ under the constraint that $E[\ln g(X)] = \alpha$, where $g(x)$ is a given pdf with support $\mathbb{R}$. Find a value for α such that the maximum differential entropy pdf is $g(x)$.