ECE 776 - Final Spring 2015

1. (2 points) Consider a binary random process that, with probability 1/3, is a stationary Markov chain with transition probabilities (from 0 to 1 and from 1 to 0) both equal to 1/3, and with probability 2/3 is a memoryless Ber(1/2) process.

a. Calculate the entropy rate.

b. Suggest a compression scheme that achieves the entropy rate.

2. (1 point) The transmitter chooses an hour of the day and the recipient receives information about the correct hour with probability 1/2 and about the previous or next hour with equal probability 1/4. No information about the day information is sent and hence the hour alphabet is {0,...,23} with 23+1=0. What is the capacity of this channel?

3. (1 point) Are the sequences x^4 =(0,0,0,1) and y^4 =(0,e,1,1) jointly typical for $\epsilon = 0.1$ if the input distribution is Ber(0.5) and the channel is an erasure channel with erasure probability 0.3? Explain.

4. (2 points) Consider a memoryless source V^m and a memoryless channel $p(y|x)$, used n times, which has capacity C. Prove that, even with feedback, the condition $n/m > H(V)/C$ is necessary to ensure vanishing probability of error as m and n go to infinity. For this purpose, use the data processing inequality, the Fano inequality and steps similar to the converse on the channel capacity.

5. (1 point) Given an i.i.d. sequence X^n of variables with zero mean and power 1, what are the probabilities of the events $\frac{1}{n}\sum_{i=1}^n X_i^2 < 1.1$ and $\frac{1}{n}\sum_{i=1}^n X_i^2 < 0.9$ when n is large?

6. (1 point) Find the maximum entropy distribution $p(x)$ with $\mathcal{X} = \{0, 1, 2\}$ and $E[X^2] = 1$.

7. (2 points) Prove that for any analog source with differential entropy h, the mean squared distortion must satisfy the inequality $D \geq \frac{2^{2h}}{2\pi e} 2^{-2R}$ if R is the number of available bits per source sample.

8. (3 points) Consider a ternary source with alphabets $\mathcal{X} = \{0, ?, 1\}$ and pmf $p(?) = q$ and $p(0) = p(1) = (1 - q)/2$

a. Calculate the entropy of this source.

b. Calculate the rate-distortion function for this source with reproduction alphabet $\widehat{\mathcal{X}} = \{0, 1\}$, under distortion metric $d(x, \hat{x})$ defined as $d(i, i) = 0$ for $i \in \{0, ?, 1\}$; $d(0, 1) = d(1, 0) = \infty$; and $d(?, i) = 0$ for $i \in \{0, 1\}$.