ECE 776 - Midterm Spring 2013

1. (1 point) For an i.i.d. sequence X^n with $Ber(1/2)$ samples, consider the sequence $L^N = (L_1, ..., L_N)$ that specifies the number of bits in the run lengths of the sequence X^n (e.g., for $X^5 = (0, 1, 1, 0, 0)$ we have $L^N = (1, 2, 2)$ and $N = 3$). Calculate $I(X^n; L^N)$.

2. (2 points) Consider an i.i.d. process $X_0, X_1, X_2, ...$ with arbitrary pmf $p(x)$ defined on a discrete alphabet $\mathcal{X}$ of finite size. Define the random variable M as the first value of $i = 1, 2, 3, ...$ such that $X_i = X_0$.

a. Calculate $E[M]$ and show that $E[M] = |\mathcal{X}|$ irrespective of $p(x)$ (Hint: Conditioning can help).

b. Show that $E[\log M] \leq H(X)$ (Hint: Conditioning is again useful and you need a famous inequality...).

3. (2 points) Consider a sequence X^n of i.i.d. variables, each taking the value 0 with probability 1/2 and 2 with probability 1/2. Define the product $M_n = X_1 X_2 ... X_n$.

a. Evaluate the pmf of M_n.

b. Evaluate mean and entropy of M_n. What happens to the entropy when $n \to \infty$?

c. How much more rate does a Shannon code need with respect to the entropy for $n = 4$?

4. (1 point) We have 3 bottles of wine, where the ith bottle is bad with probability p_i, independently of all the other bottles. In other words, if $X_i = 1$ when the bottle is bad and $X_i = 0$ otherwise, then the variables X_1, X_2, X_3 are independent with $Pr[X_i = 1] = p_i$. We are asked to determine the set of all bad bottles. Any binary question is admissible for this purpose. Give good lower and upper bounds on the minimum average number of questions required.

5. (1 point) Consider the Bernoulli distribution with $p = 0.1$. Is it true that $(0, 0, 1, 0, 0) \in A_\epsilon^{(5)}$ for $\epsilon = 0.1$? If not, what is the smallest set $A_\epsilon^{(5)}$ (for the given distribution) that includes this sequence?

6. (1 point) Consider the random walk process defined as $S_0 = 0$ and $S_n = S_{n-1} + X_n$ for $n \geq 1$, where $X_1, X_2, X_3, ...$ are i.i.d. with pmf given as $p_X(1) = 1/2$ and $p_X(2) = 1/2$.

a. Is the process $\{S_n\}$ stationary?

b. Calculate $H(S_1, S_2, ..., S_n)$.

c. Does the process $\{S_n\}$ have an entropy rate? If so, what is it? If not, why not?

7. (1 point) Consider the two-state Markov chain $Y_1, Y_2, ..$ with transition probabilities $p(1|0) = 1/3$ and $p(0|1) = 1/3$.

a. Which initial distribution guarantees the stationarity of the Markov chain? Calculate the pmf $\{p_Y(0), p_Y(1)\}$ under this initial distribution.

b. Consider now another process $X_1, X_2, ...$ which is obtained from $Y_1, Y_2, ..$ as follows. If $Y_i = 1$, then X_i has pmf $p_{X|Y}(x|1)$ uniform in the set $\{1, 2, 3, 4\}$. Instead, if $Y_i = 0$, then X_i has pmf $p_{X|Y}(x|0)$ uniform in the set $\{5, 6, 7, 8\}$. Calculate the entropy rate of $X_1, X_2, ...$

8. (2 points) Consider a random variable X that takes on m values $\{1, 2, ..., m\}$ with probabilities $p_1, p_2, ..., p_m$, which are ordered as $p_1 \geq p_2 \geq \cdots \geq p_m$. Consider the following variation of Shannon-Fano-Elias coding, in which the code for the ith symbol is given as $l_i = \lceil -\log_2 p_i \rceil$ and $c_i = \lfloor F(i-1) \rfloor_{l_i}$, where $F(i) = \sum_{j=1}^{i} p_j$ is the cumulative mass function.

a. Show that the code constructed by this process is prefix-free.

b. Show that the average length satisfies $H(X) \leq L < H(X) + 1$. Compare with standard Shannon-Fano-Elias coding.

c. Construct the code for the probability distribution $(0.5, 0.25, 0.125, 0.125)$.