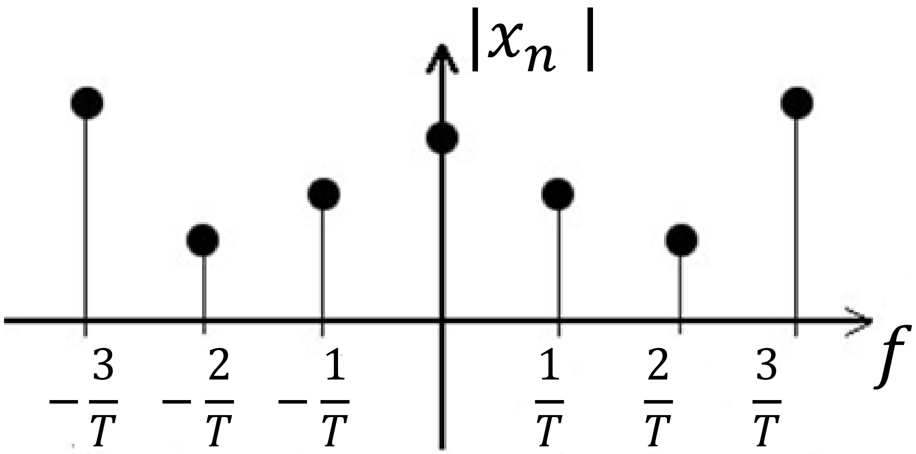
Frequency-domain Analysis: Energy Signals

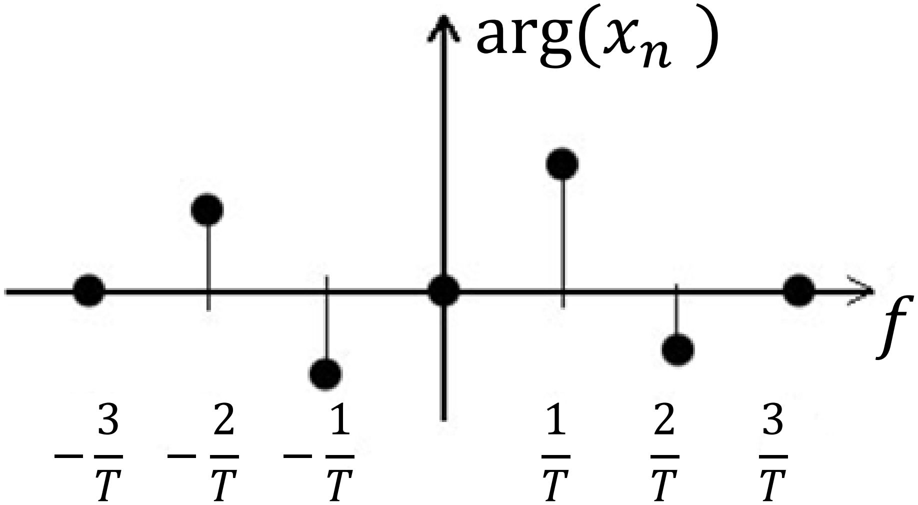
(Chapter 2: 2.2)

- The Fourier transform represents an energy signal as the infinite sum of "oscillations" in a (continuous) spectrum of frequencies.

Frequency-domain Analysis: Energy Signals

- Fourier series – periodic signals

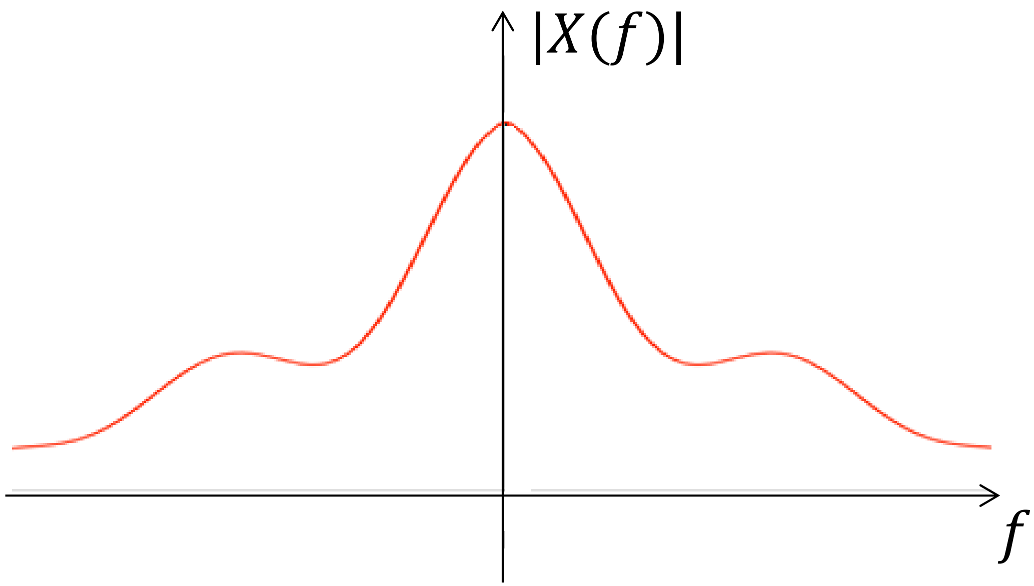


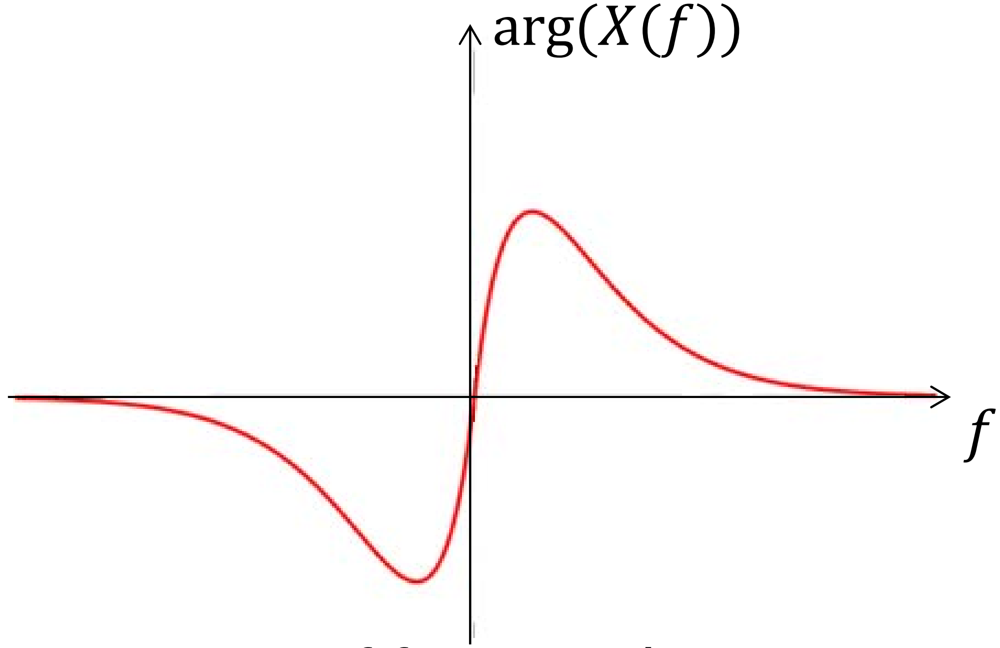


sum of harmonics (multiple of $\frac{1}{T}$)

Frequency-domain Analysis: Energy Signals

- Fourier transform – energy signals





sum over a continuous spectrum of frequencies

Frequency-domain Analysis: Energy Signals

- Fourier transform

inverse Fourier transform

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j\,2\pi f t} df = \mathcal{F}^{-1}\{X(f)\}$$

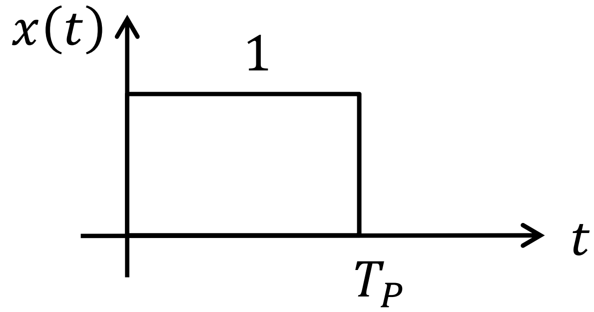
and

Fourier transform

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\,2\pi f t} dt = \mathcal{F}\{x(t)\}$$

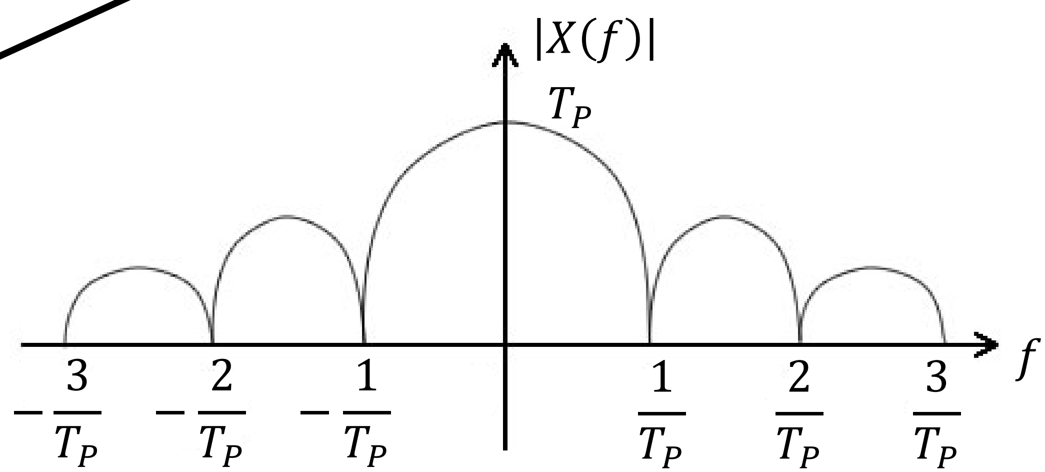
Example a)

$$x(t) = \begin{cases} 1 & 0 \le t \le T_P \\ 0 & \text{elsewhere} \end{cases}$$



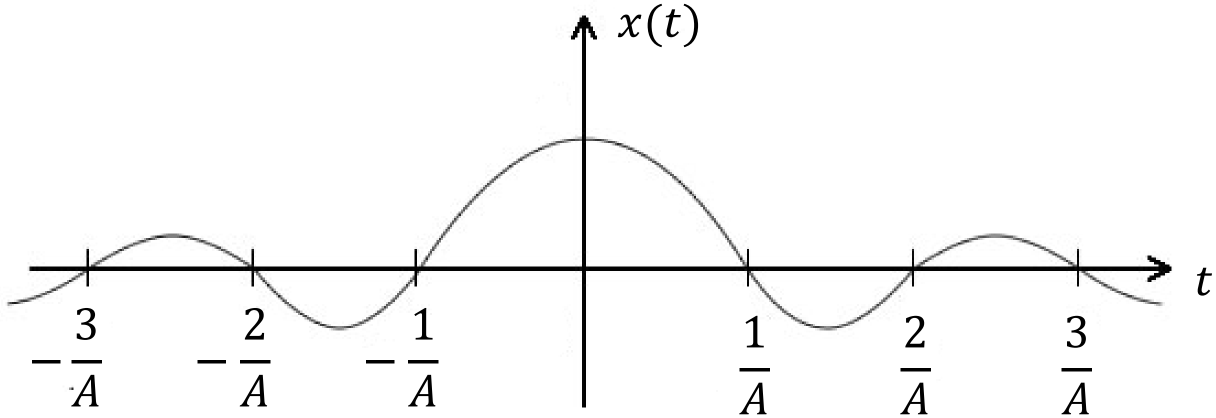
$$X(f) = \int_0^{T_P} e^{-j\,2\pi f t}\,dt = T_P\, sinc(f\,T_P) e^{-j\,\pi f T_P}$$

See textbook

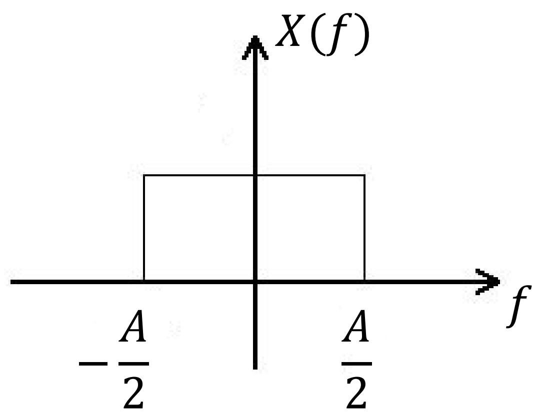


Example b)

$$x(t) = A\,\mathrm{sinc}(At)$$



$$X(f) = \begin{cases} 1 & |f| \le \frac{A}{2} \\ 0 & \text{elsewhere} \end{cases}$$



Properties of the Fourier Transform

1) If $x(t)$ is real

$$\left.\begin{array}{l} X(f) = X^*(-f) \quad \text{or equivalently:} \\ \begin{cases} |X(f)| = |X(-f)| \\ \arg(X(f)) = -\arg(X(-f)) \end{cases} \end{array}\right\} \text{Hermitian symmetry}$$

Ex.: Rectangular function, sinc

Properties of the Fourier Transform

2) If $x(t)$ is real and even (i.e., $x(t) = x(-t)$)

$\longrightarrow X(f)$ is real and even ($X(f) = X(-f)$)

Ex.: sinc, rectangular function centered at $t = 0$

3) If $x(t)$ is real and odd (i.e., $x(t) = -x(-t)$)

$\longrightarrow X(f)$ is real and odd (i.e., $X(f) = -X(-f)$))

Properties of the Fourier Transform

4) Rayleigh theorem

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X(f)|^2 \, df$$

Remark: Energy can be calculated both in time and frequency domains

Ex.: $x(t) = \mathrm{A}\, sinc(At) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \begin{cases} 1 & |f| \leq \frac{A}{2} \\ 0 & \text{elsewhere} \end{cases}$

$$E_x = A$$

$\blacksquare$

Properties of the Fourier Transform

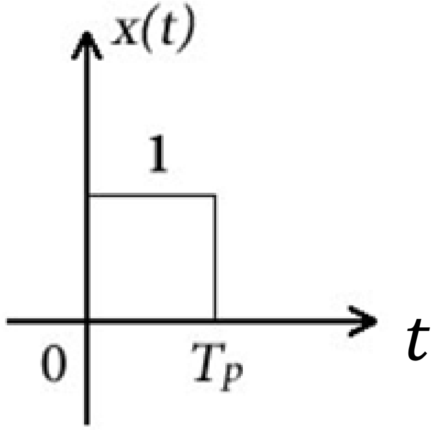
5) Delay

$$\mathcal{F}\{x(t-\tau)\} = X(f)\, e^{-j\,2\pi f \tau}$$

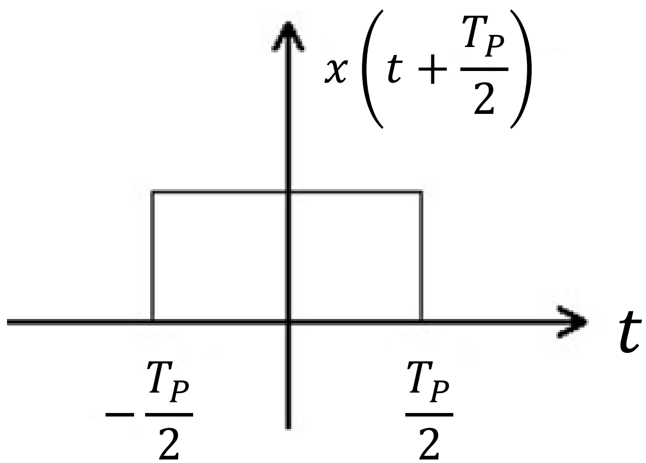
Remark: A delay in time domain causes a linear phase shift in the frequency domain

Properties of the Fourier Transform

Ex.:



$$X(f) = T_P sinc(fT_P)e^{-j\pi f T_P}$$



$$\mathcal{F}\left\{x\left(t+\frac{T_P}{2}\right)\right\} = T_P\, sinc(fT_P)$$

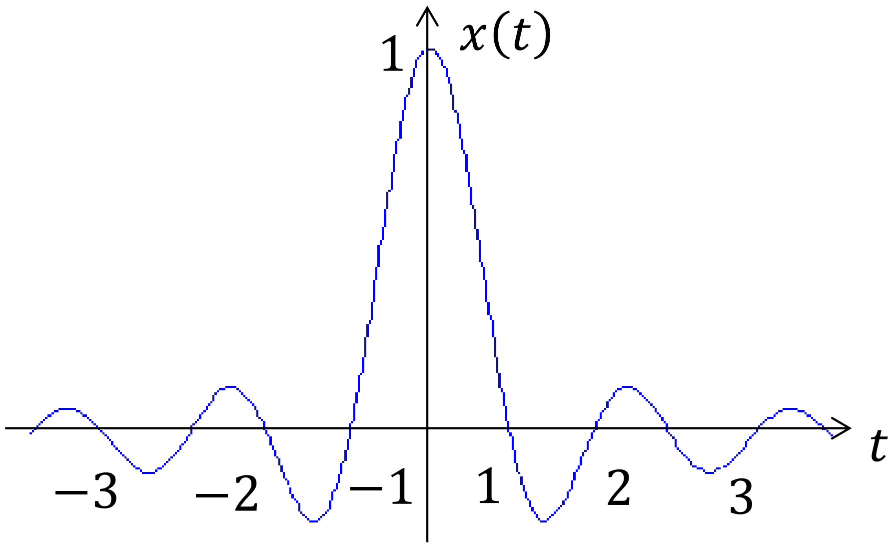
Properties of the Fourier Transform

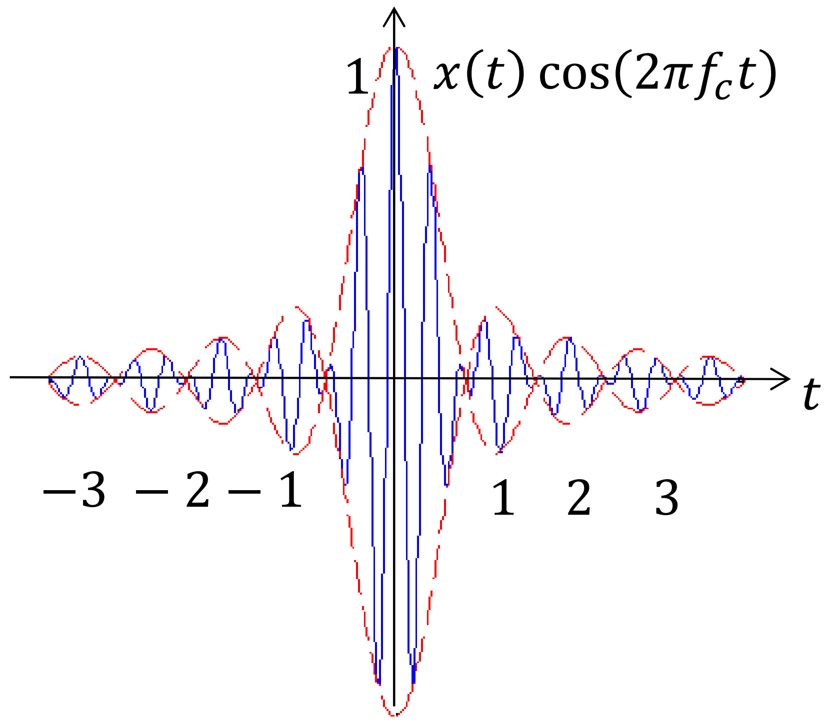
6) Frequency translation

$$\mathcal{F}\{x(t)e^{j\,2\pi f_c t}\} = X(f - f_c)$$

Ex.: $x(t) = \mathrm{sinc}(t)$

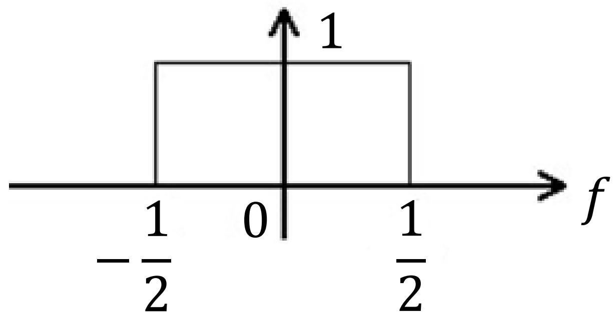
$$x(t)e^{j\,2\pi f_c t} = x(t)\cos(2\pi f_c t) + jx(t)\sin(2\pi f_c t)$$



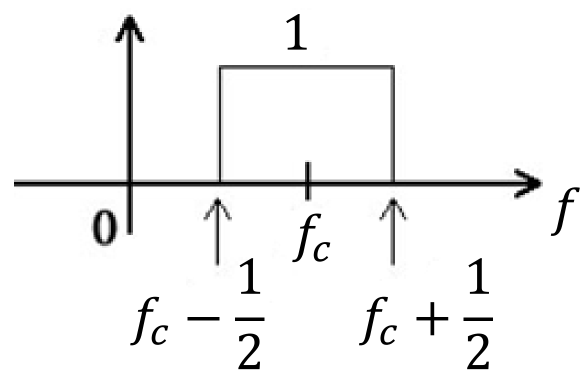


Properties of the Fourier Transform

$X(f) =$



$X(f - f_c) =$



Remark: The operation of multiplying by $e^{j\,2\pi f_c t}$ is called "upconversion" in communication systems.

Properties of the Fourier Transform

- Additional facts to know about the Fourier transform

a) Pages 2.10-2.11: further properties

b) Pages 2.14-2.16: defining the bandwidth of a signal

Energy Spectrum and Correlation Function

- Energy spectrum of a signal $x(t)$

$$G_x(f) = |X(f)|^2$$

energy spectral density $[J/Hz]$

- Measures the energy of the signal at frequency f.

Energy Spectrum and Correlation Function

- Integrating across all frequencies, we obtain energy of the signal:

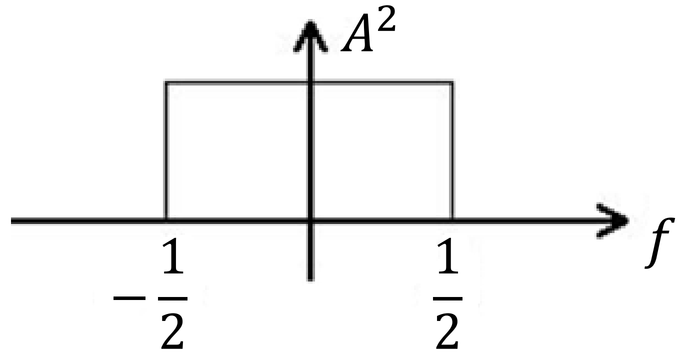
$$E_x = \int_{-\infty}^{+\infty} G_x(f)\, df$$

$$= \int_{-\infty}^{+\infty} |X(f)|^2\, df$$

Energy Spectrum and Correlation Function

Ex.: $x(t) = Asinc(t)$

$G_x(f) =$



$E_x = A^2$

Energy Spectrum and Correlation Function

- Correlation function of a signal $x(t)$

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t-\tau)dt$$

- Measures the correlation between $x(t)$ and $x(t-\tau)$ (i.e., $x(t)$ delayed by τ)

- $\mathcal{F}\{R_x(\tau)\} = G_x(f)$

Energy Spectrum and Correlation Function

- Properties of the correlation function

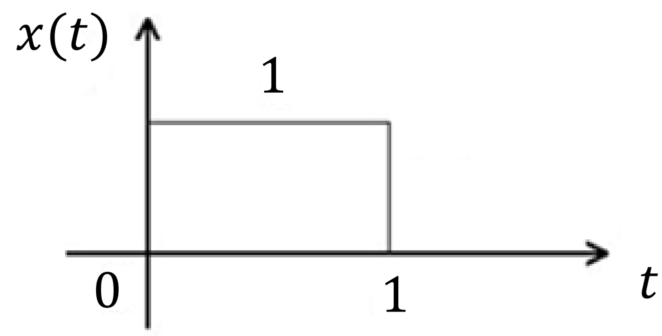
$$R_x(0) = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = E_x$$

$R_x(\tau) = R_x^*(-\tau)$ Hermitian symmetry

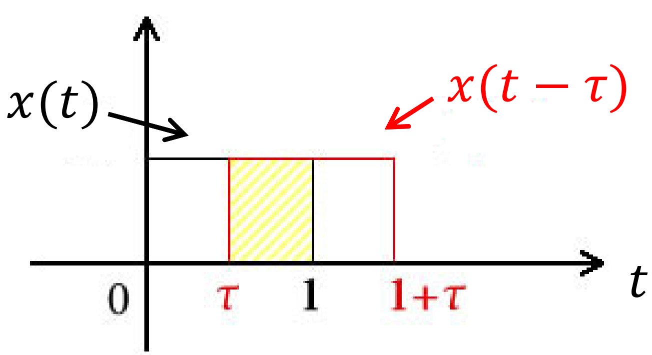
$$|R_x(\tau)| < R_x(0) \quad \text{for } \tau \neq 0$$

Energy Spectrum and Correlation Function

Ex.: $x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$



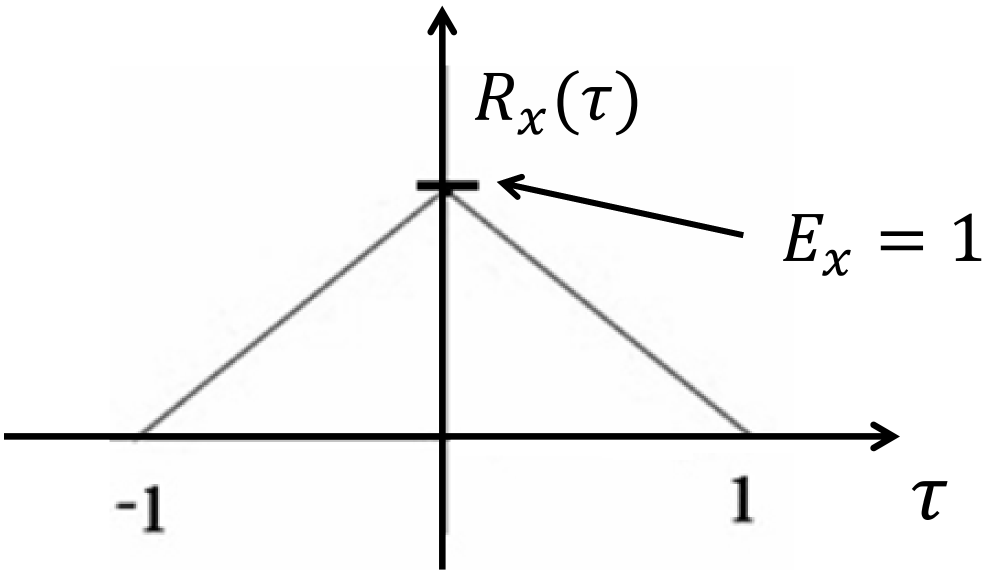
Calculating $R_x(\tau)$:



Area marked in yellow

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t-\tau)dt = \begin{cases} 1 - |\tau| & \text{for } |\tau| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Energy Spectrum and Correlation Function



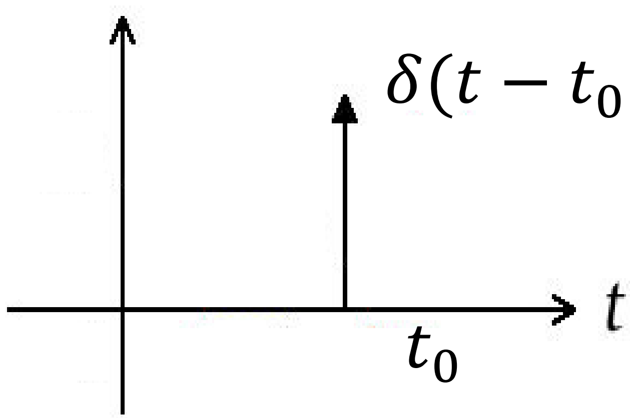
- All properties above are satisfied (check!)
- Correlation decreases linearly as $|\tau|$ increases

Fourier Series as a Fourier Transform

- The Fourier transform has been defined only for energy signals and hence does not apply to periodic signals. Can we extend this definition to include also the Fourier series as a special case?

Fourier Series as a Fourier Transform

- Impulse function:



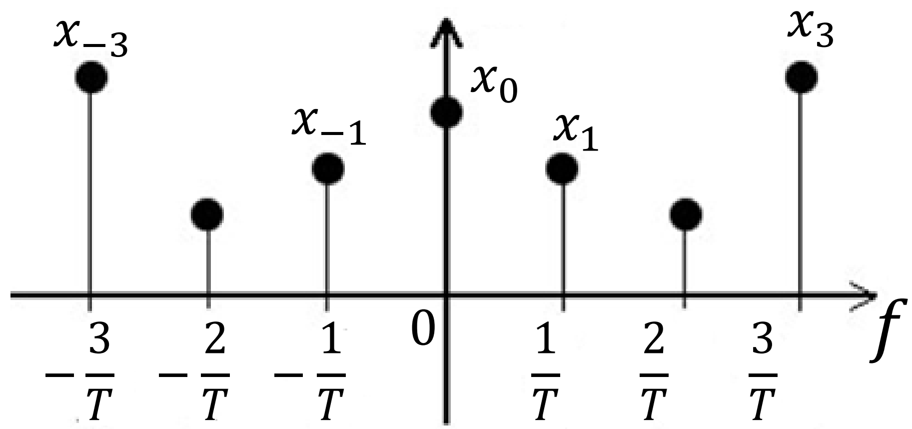
$$\begin{cases} \delta(t - t_0) = 0 \quad \text{for all } t \neq t_0 \\ \int_{-\infty}^{+\infty} f(t)\,\delta(t - t_0)\,dt = f(t_0) \end{cases}$$

Fourier Series as a Fourier Transform

- Fourier series

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n e^{j2\pi \frac{n}{T} t}$$

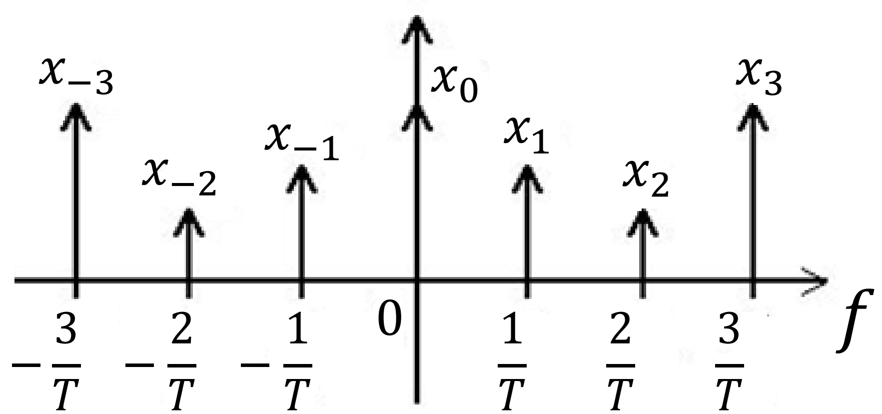
Periodic signal with period T



Fourier Series as a Fourier Transform

- For a periodic signal $x(t)$, define the Fourier transform as

$$X(f) = \sum_{n=-\infty}^{+\infty} x_n \, \delta\left(f - \frac{n}{T}\right)$$



Fourier Series as a Fourier Transform

- With this definition, we get

$$x(t) = \mathcal{F}^{-1}\{X(f)\}$$

$$= \int_{-\infty}^{+\infty} X(f) e^{j2\pi f t}\, df$$

$$= \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x_n\, \delta\left(f - \frac{n}{T}\right) e^{j2\pi f t}\, df$$

continued on next slide…

Fourier Series as a Fourier Transform

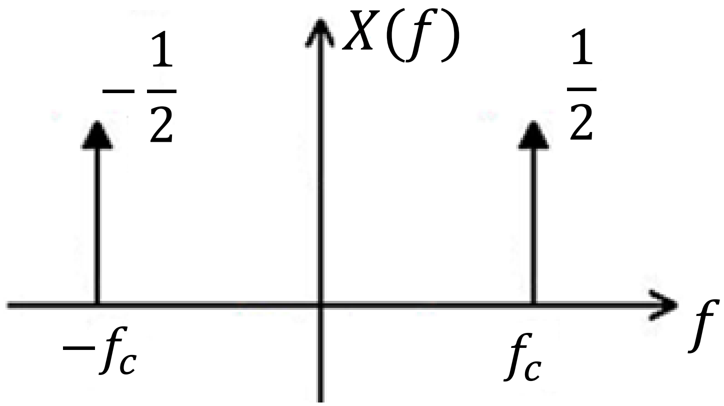
- With this definition, we get

$$= \sum_{n=-\infty}^{+\infty} x_n \left(\int_{-\infty}^{+\infty} \delta\left(f - \frac{n}{T}\right) e^{j2\pi f t} \right) df$$

$$= \sum_{n=-\infty}^{+\infty} x_n \, e^{j2\pi \frac{n}{T} t}$$ which coincides with Fourier series

Examples

a) $x(t) = \cos(2\pi f_c t)$



b) See Ex. 2.27 for train of rectangular pulses