NJIT

New Jersey's Science & Technology University

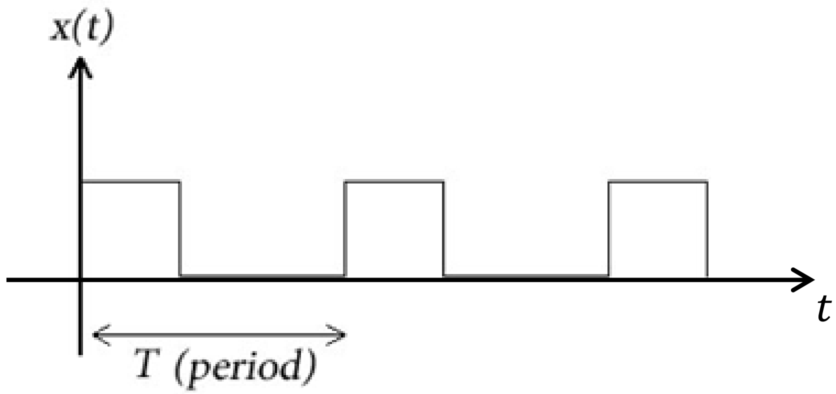
THE EDGE IN KNOWLEDGE

Frequency-domain Analysis: Periodic Signals

(Chapter 2: 2.2)

- The Fourier series represents a periodic signal as the sum of “oscillations” of frequencies multiple of the fundamental frequency $\frac{1}{T}$

Frequency-domain Analysis: Periodic Signals



periodic signal

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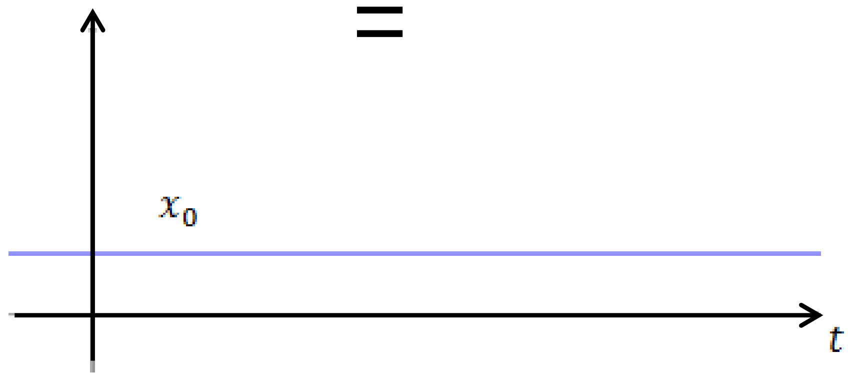

zero frequency ⟵ 0th harmonic

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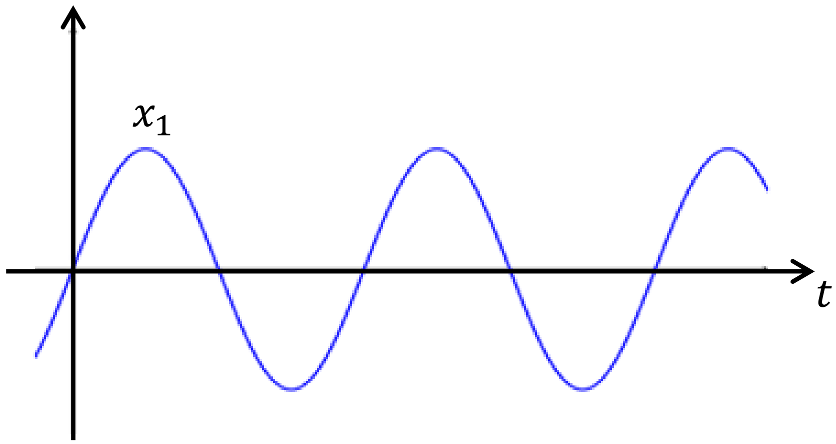
Frequency-domain Analysis: Periodic Signals



Frequency $\frac{1}{T}$ ⟵ 1^{st} harmonic

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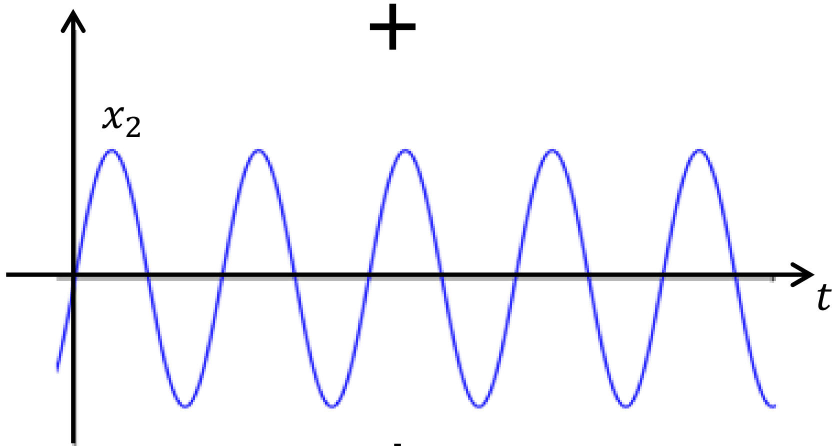

Frequency $\frac{2}{T}$ ⟵ 2^{nd} harmonic

⋮

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...

Mathematical Definition of Fourier Series

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n \, e^{j\,2\pi \frac{n}{T}t}$$

$$= \cdots + x_{-2}e^{-j\,2\pi \frac{2}{T}t} + x_{-1}e^{-j\,2\pi \frac{1}{T}t} + x_0$$

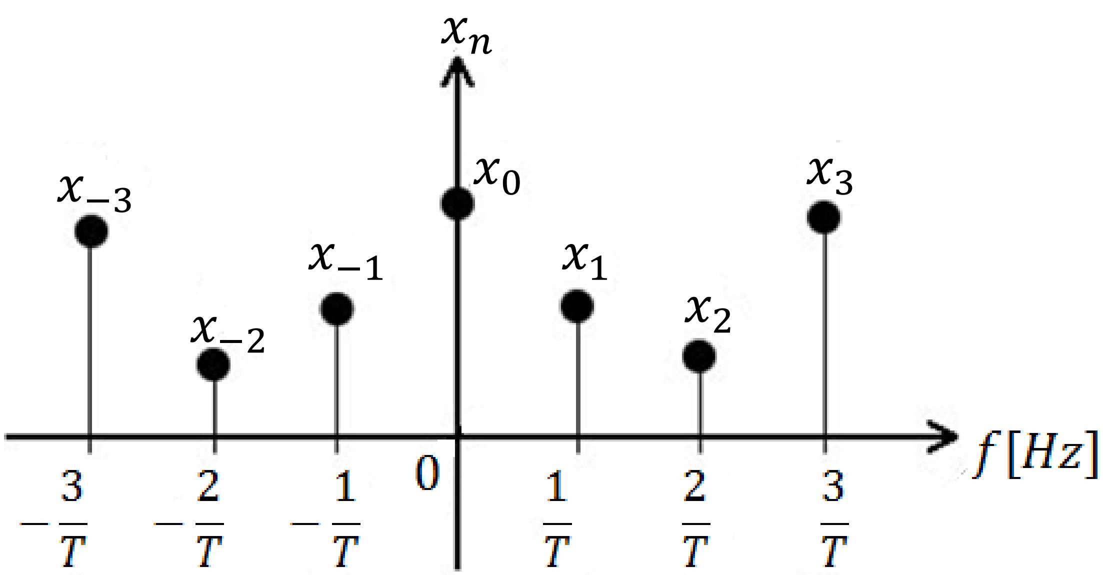
$$+x_1 e^{j\,2\pi \frac{1}{T}t} + x_2 e^{j\,2\pi \frac{2}{T}t} + \cdots$$

Remark: $e^{j\,2\pi \frac{n}{T}t}$ – "oscillation" of frequency $\frac{n}{T}$

Recall that $e^{j\,2\pi \frac{n}{T}t} = \cos\left(\frac{2\pi n t}{T}\right) + j\sin\left(\frac{2\pi n t}{T}\right)$

Fourier Series

- Thanks to the Fourier series we can represent a periodic signal in the frequency domain as



Fourier Series

- How to calculate the coefficients x_n of the Fourier series?

$$x_n = \frac{1}{T}\int_0^T x(t)e^{-j\,2\pi\frac{n}{T}t}\,dt \quad n = \cdots, -2, -1, 0, 1, 2, \ldots$$

Remark: The integral above calculates the correlation between $x(t)$ and the "oscillation" at the n^{th} harmonics $\frac{n}{T}$

$\longrightarrow$ x_n measures how "similar" $x(t)$ and $e^{j\,2\pi\frac{n}{T}t}$ are.

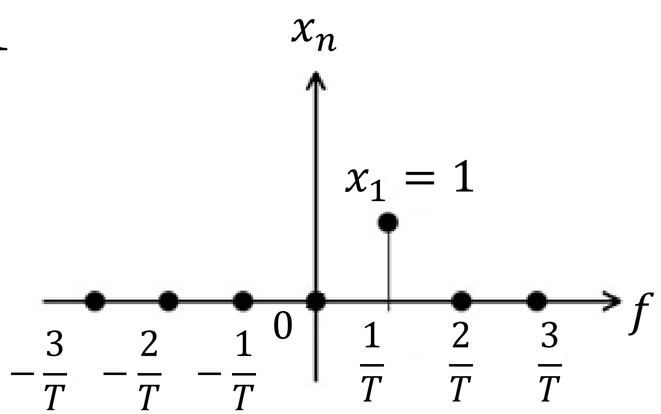
Example a)

$x(t) = e^{j\,2\pi f_c t} = \cos(2\pi f_c t) + j\sin(2\pi f_c t)$

periodic signal with period $T = \frac{1}{f_c}$

Fourier series:

$x_1 = 1$ and $x_n = 0$ for all $n \neq 1$



In fact,

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n\, e^{j\,2\pi \frac{n}{T} t} = \sum_{n=-\infty}^{+\infty} x_n\, e^{j\,2\pi n f_c t} = e^{j\,2\pi f_c t}$$

Example b)

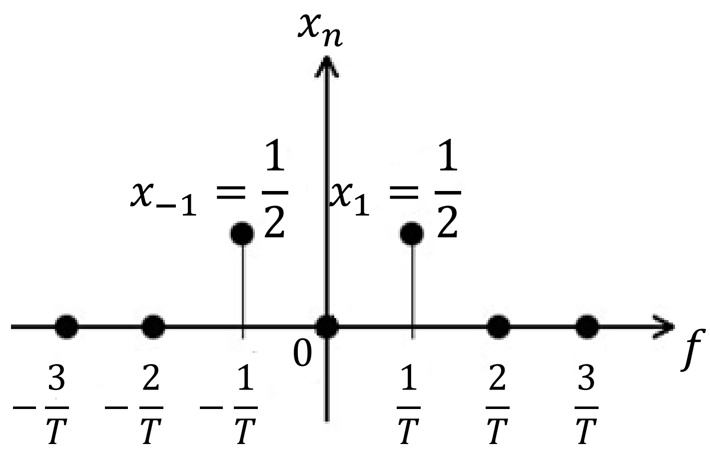
$x(t) = \cos(2\pi f_c t)$

Periodic signal with period $f_c = \frac{1}{T}$

Fourier series

$x_1 = \frac{1}{2}, \quad x_{-1} = \frac{1}{2},$

and $x_n = 0$ for $n \neq \pm 1$

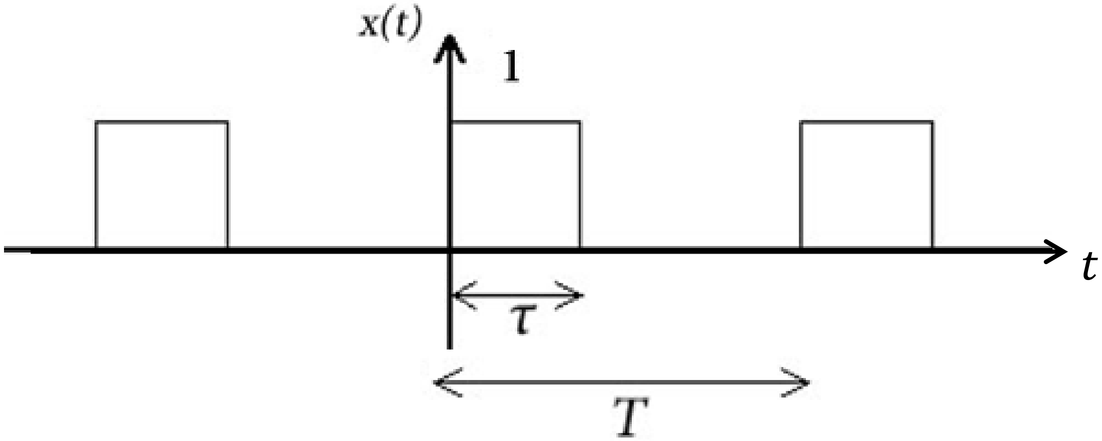


This follows from

$\cos(2\pi f_c t) = \frac{1}{2} e^{j\,2\pi f_c t} + \frac{1}{2} e^{-j\,2\pi f_c t}$ Euler's formula

Example c)

Pulse train



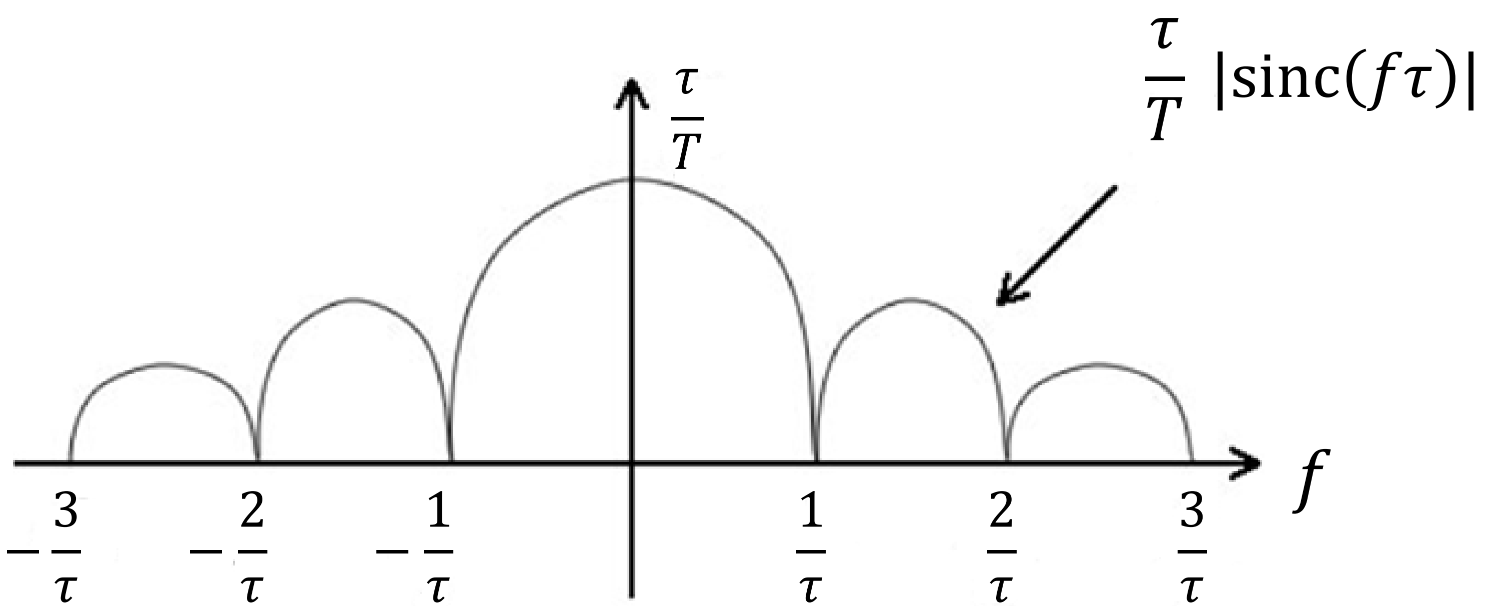
Periodic signal with period T

Fourier series

$$x_n = \frac{\tau}{T}\operatorname{sinc}\left(\frac{n}{T}\tau\right) e^{-j\pi\frac{n}{T}t}$$

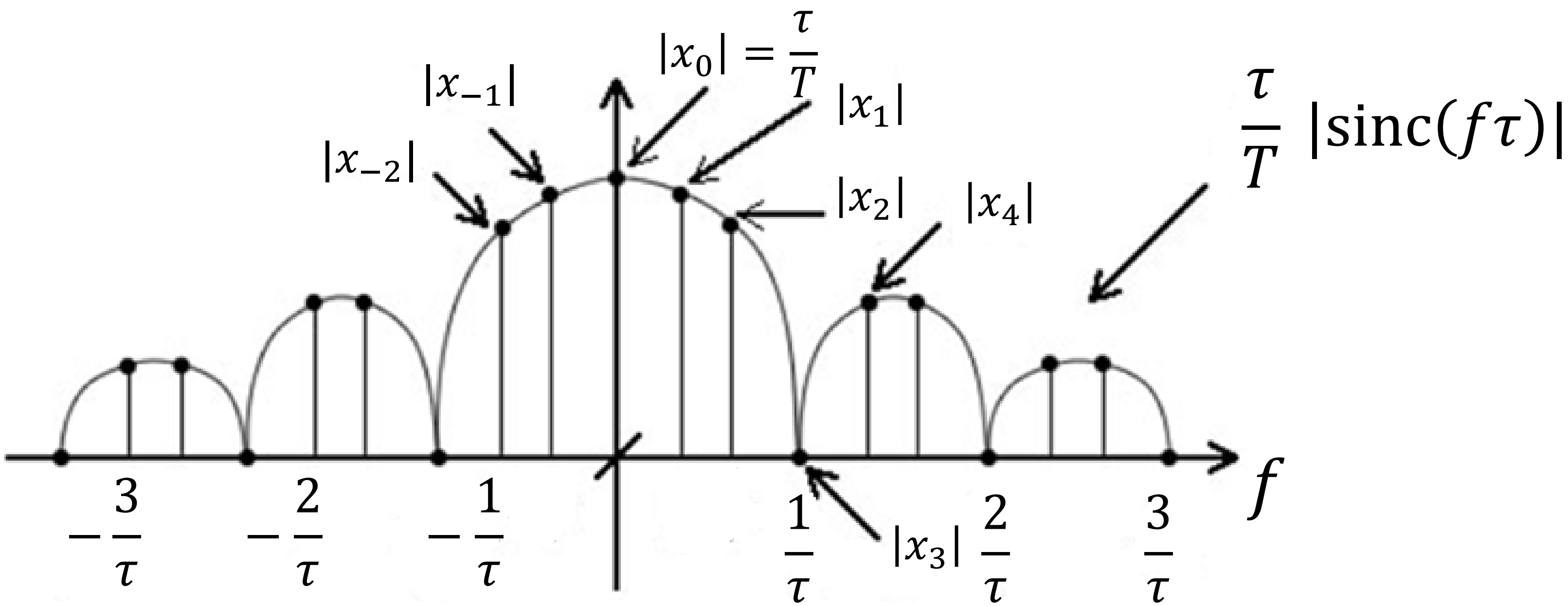
Example c)

- Let us plot it for $\tau = \frac{T}{3}$



Example c)

- Let us plot it for $\tau = \frac{T}{3}$ $\left(\frac{1}{T} = \frac{1}{3} \cdot \frac{1}{\tau}\right)$



Example c)

- For derivation, please see the textbook
- If the signal duration τ increases, the "bandwidth" $\frac{1}{\tau}$ decreases
- Infinite number of harmonics to produce the sharp edges of a rectangle
- Try the applet: http://www.falstad.com/fourier/

(Best viewed in Internet Explorer or Chrome)

Properties of the Fourier Series

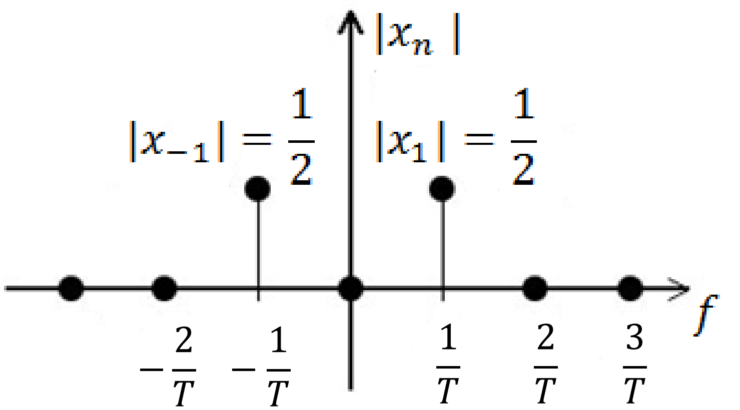
1) If $x(t)$ is real

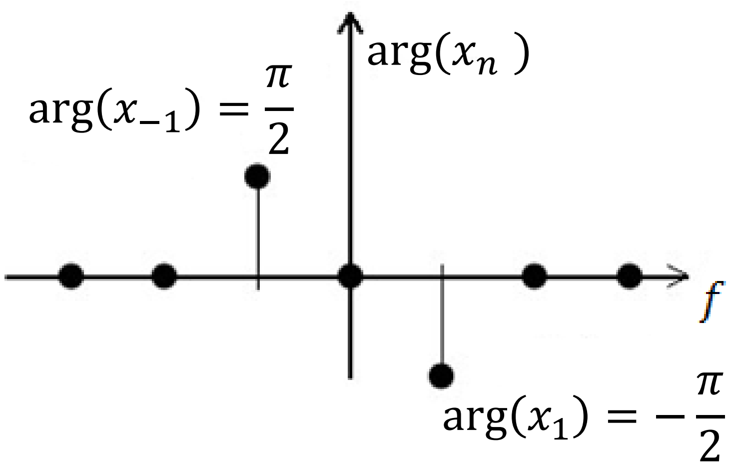
$x_n = x^*_{-n}$ or equivalently:

$$\begin{cases} |x_n| = |x_{-n}| \\ \arg(x_n) = -\arg(x_{-n}) \end{cases}$$

Hermitian symmetry

Ex: $x(t) = \sin\left(\frac{2\pi t}{T}\right)$





Properties of the Fourier Series

2) If $x(t)$ is real and even (i.e., $x(t) = x(-t)$)

$\longrightarrow$ x_n is real and even (i.e., $x_n = x_{-n}$)
Ex.: Cosine

3) If $x(t)$ is real and odd (i.e., $x(t) = -x(-t)$)

$\longrightarrow$ x_n is imaginary and odd (i.e., $x_n = -x_{-n}$)
Ex.: Sine

Properties of the Fourier Series

4) Parseval theorem

$$P_x = \frac{1}{T}\int_0^T |x(t)|^2\, dt = \sum_{n=-\infty}^{+\infty} |x_n|^2$$

Remark: Power can be calculated both in time and in frequency domains

Ex.: For $x(t) = \cos(2\pi f_c t)$, the power is

$$P_x = \int_0^T |x(t)|^2 dt = \frac{1}{2} = |x_{-1}|^2 + |x_1|^2$$

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