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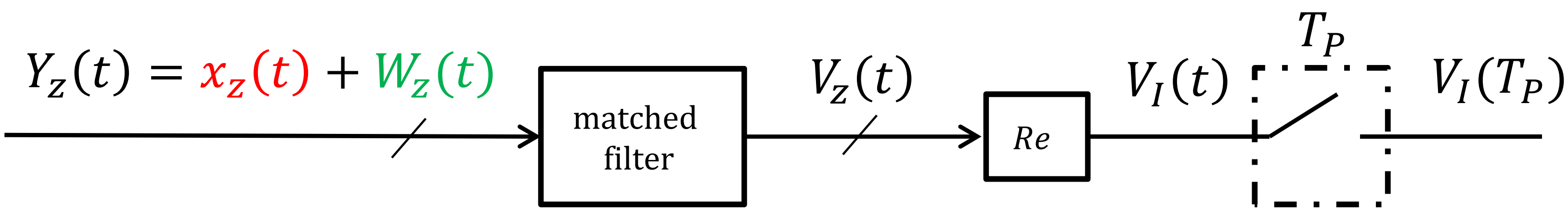
THE EDGE IN KNOWLEDGE

- Consider now the effect of the matched filter on the sufficient statistics $V_I(T_P)$:



- Recall that we have $V_I(T_P) = m_{0/1} + N_I$

if $M = 0$ if $M = 1$

where $m_{0/1} = Re\{x_{z,0/1}(t) * h(t)\big|_{t=T_P}\}$

$$N_I \sim N(0, \sigma_{N_I}^2) \quad \text{with } \sigma_{N_I}^2 = \frac{N_0}{2}\int_{-\infty}^{+\infty} |H(f)|^2 \, df$$

Let's calculate $\sigma_{N_I}^2$ first:

$$\sigma_{N_I}^2 = \frac{N_0}{2}\int_{-\infty}^{+\infty} |H(f)|^2\, df$$

$$= \frac{N_0}{2}\int_{-\infty}^{+\infty} |h(t)|^2\, dt$$

Rayleigh theorem

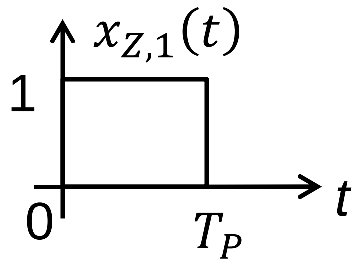
$$= \frac{N_0}{2}\int_{-\infty}^{+\infty} \left|x_{z,1}^*(-t+T_P) - x_{z,0}^*(-t+T_P)\right|^2 dt$$

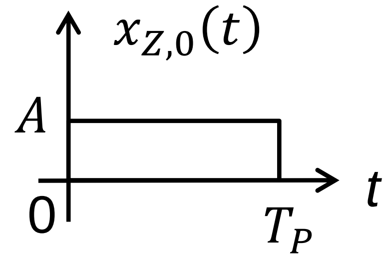
$$= \frac{N_0}{2}\int_{-\infty}^{+\infty} \left|x_{z,1}(t) - x_{z,0}(t)\right|^2 dt = \frac{N_0}{2}\Delta_E(1,0)$$

Squared Euclidean distance between $x_{z,1}(t)$ and $x_{z,0}(t)$: $\Delta_E(1,0)$

- Remark: The Euclidean distance $\Delta_E(1,0)$ measures how “far” $x_{z,1}(t)$ and $x_{Z,0}(t)$ are from each other.

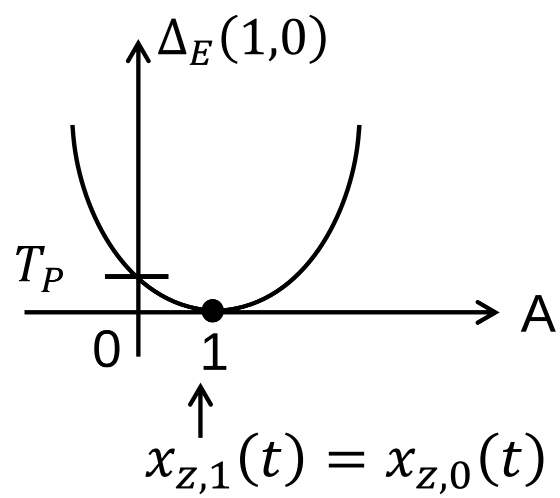
Ex.:





$$\Delta_E(1,0) = \int_{-\infty}^{+\infty} \left|x_{Z,1}(t) - x_{Z,0}(t)\right|^2 dt = \int_0^{T_P} (1-A)^2 dt$$

$$= (1-A)^2 T_P$$



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- Remark: $\Delta_E(1,0) = \Delta_E(0,1)$

- Remark: The squared Euclidean distance can be expressed in terms of the energies $E_o$and $E_1$and in terms of the correlation coefficient:

$$\rho_{10} = \frac{\int_{-\infty}^{+\infty} x_{z,1}(t) x_{z,0}^{*}(t) dt}{\sqrt{E_0 \, E_1}}$$

- In fact:

$$\Delta_E(1,0) = \int_{-\infty}^{+\infty} \left(x_{z,1}(t) - x_{z,0}(t)\right)^{*} \left(x_{z,1}(t) - x_{z,0}(t)\right) dt$$

$$= \int_{-\infty}^{+\infty} \left|x_{z,1}(t)\right|^2 dt + \int_{-\infty}^{+\infty} \left|x_{z,0}(t)\right|^2 dt - 2Re\left\{\int_{-\infty}^{+\infty} x_{z,1}(t) x_{z,0}^{*}(t) dt\right\}$$

$$= E_1 + E_0 - 2\sqrt{E_0 \, E_1} \, Re(\rho_{10})$$

Ex.: With the waveforms of the previous example, we have:

$$E_1 = T_P, \quad E_0 = A^2 T_P$$

$$\rho_{10} = \frac{\int_{-\infty}^{+\infty} x_{z,1}(t) x_{z,0}^*(t) dt}{\sqrt{E_0 \, E_1}}$$

$$= \frac{\int_0^{T_P} A dt}{\sqrt{A^2 {T_p}^2}} = \frac{AT_p}{|A| T_p} = \mathrm{sign}(A) = \begin{cases} +1 \text{ if } A > 0 \\ 0 \quad \text{if } A = 0 \\ -1 \text{ if } A < 0 \end{cases}$$

$$\Rightarrow \Delta_E(1,0) = E_1 + E_0 - 2sign(A)\sqrt{E_0 \, E_1}$$

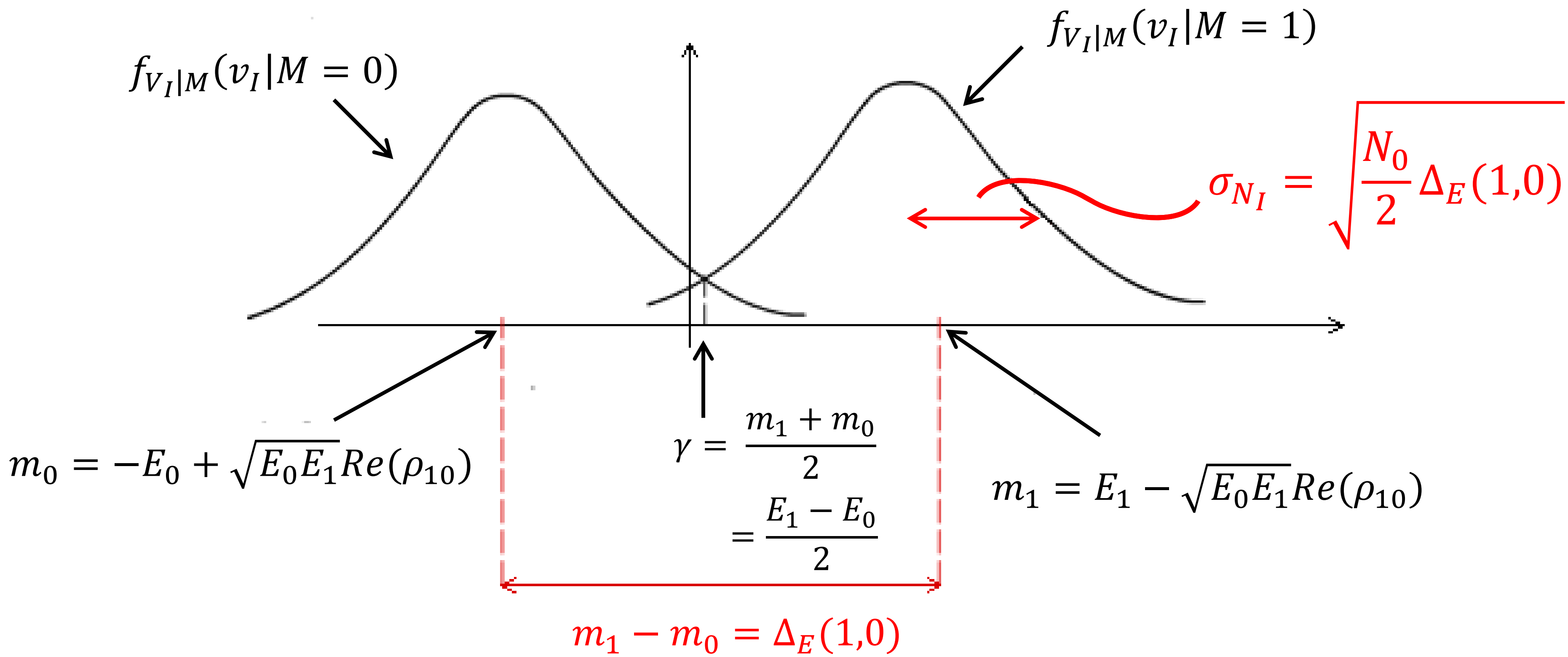
$$= (1 + A^2) T_P - 2AT_P = (1 - A)^2 T_P$$

- Using similar calculations, we get

$$m_1 = E_1 - \sqrt{E_0\ E_1} Re(\rho_{10})$$

$$m_0 = -E_0 + \sqrt{E_0\ E_1} Re(\rho_{10})$$

To summarize:



Effective SNR:

$$\eta = \frac{(m_1-m_0)^2}{8\sigma_{N_I}^2} = \frac{\Delta_E(1,0)}{4N_0}$$

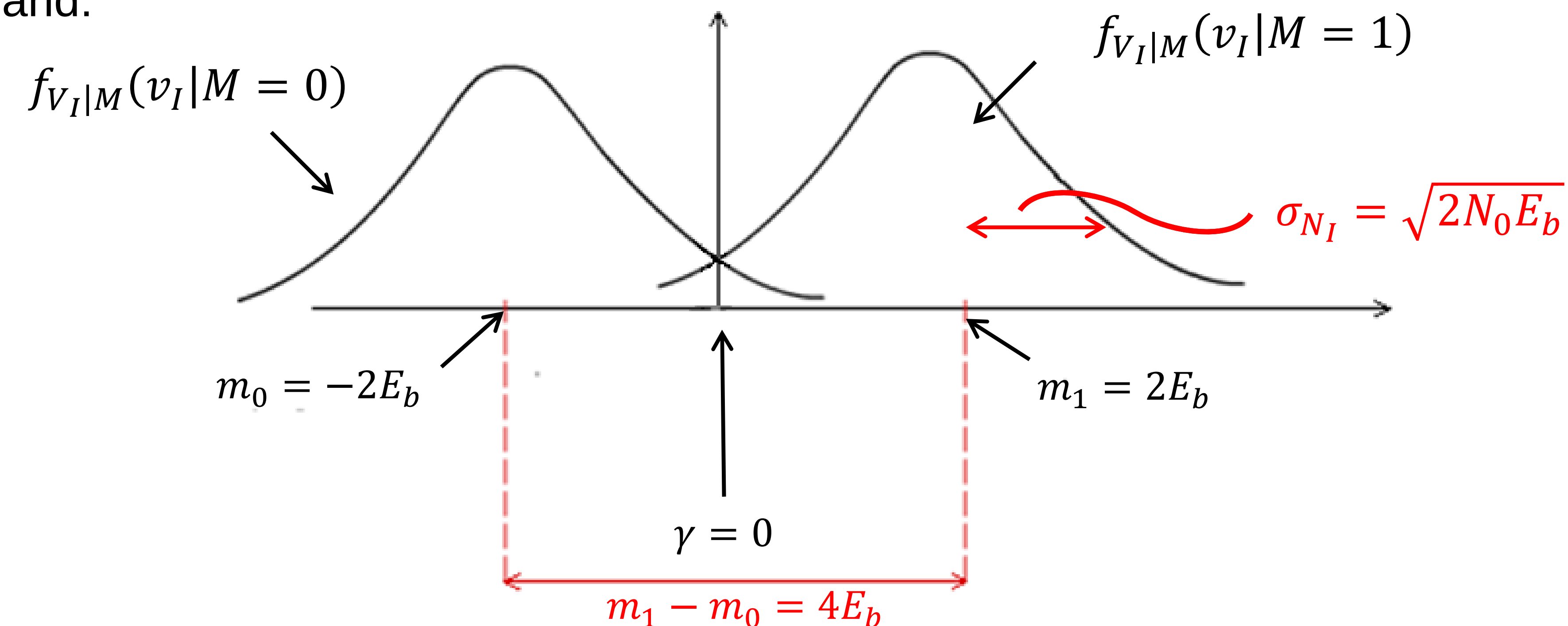
Increasing $\Delta_E(1,0)$ improves η.

In the special case in which $x_{z,1}(t) = -x_{z,0}(t)$, we have:

$$E_1 = E_0 = E_b$$

$$\Delta_E(1,0) = 4E_b \text{ (since } \rho_{10} = -1)$$

and:



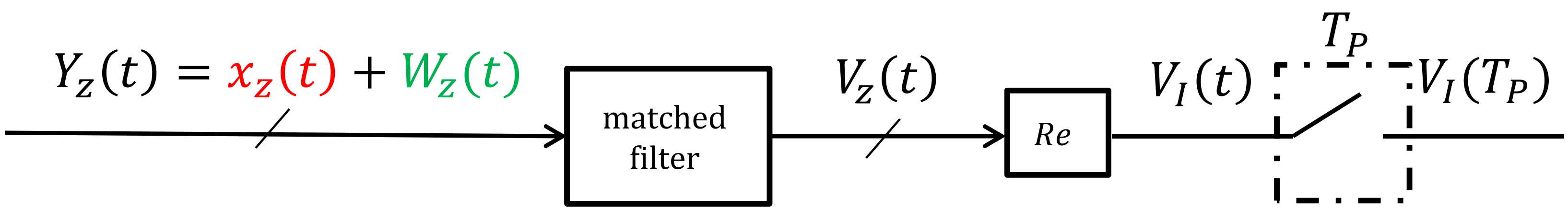
Effective SNR:

$$\eta = \frac{(m_1 - m_0)^2}{8\sigma_{N_I}^2} = \frac{E_b}{N_0}$$

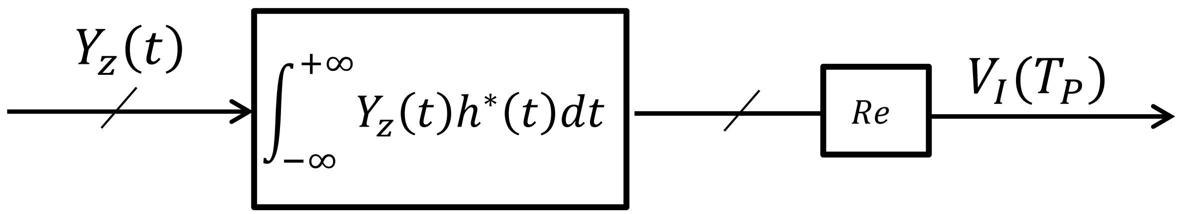
- We have seen that the filter matched to $x_{z,i}(t)$ for $i = 0,1$ can be implemented as a correlator
- Since the matched filter is

$H(f) = \left(X_{z,1}^*(f) - X_{z,0}^*(f)\right) e^{-j2\pi f T_p}$ and $h(t) = x_{z,1}^*(-t + T_P) - x_{z,0}^*(-t + T_P)$

we have that the structure "matched filter + Re + sampler"

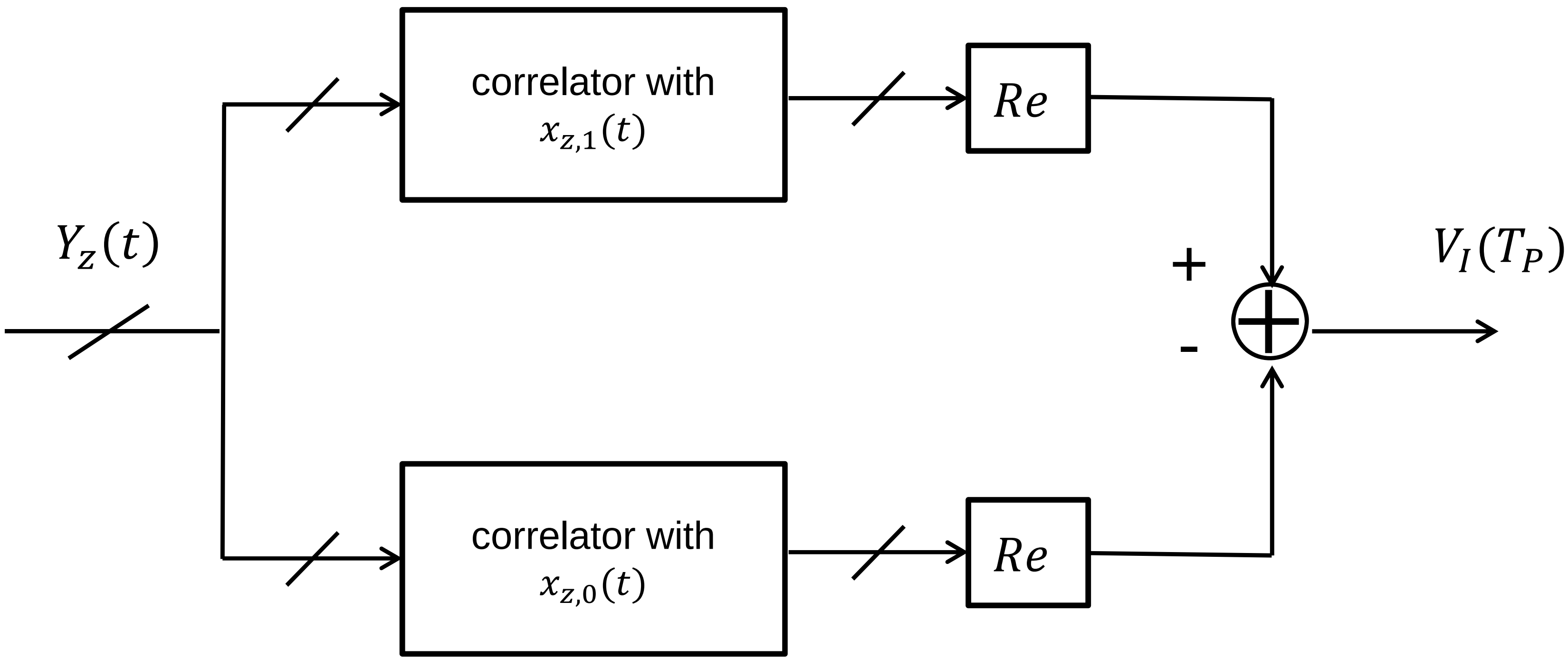


is equivalent to the "correlator + Re" implementation:



correlator with $h(t)$

Bank of correlators implementation:



- Remark: The correlator + Re implementation is often more convenient, especially for digital signal processing

- Note that

$$V_I(T_P) \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} \gamma = \frac{E_1 - E_0}{2}$$

is equivalent to

$$V_I(T_P) = \underbrace{Re\left\{\int_{-\infty}^{+\infty} Y_z(t)x_{z,1}^*(t)dt\right\}}_{\text{correlator with } x_{z,1}(t)} - \underbrace{Re\left\{\int_{-\infty}^{+\infty} Y_z(t)x_{z,0}^*(t)dt\right\}}_{\text{correlator with } x_{z,0}(t)} \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} \gamma$$

which is in turn equivalent to the rule

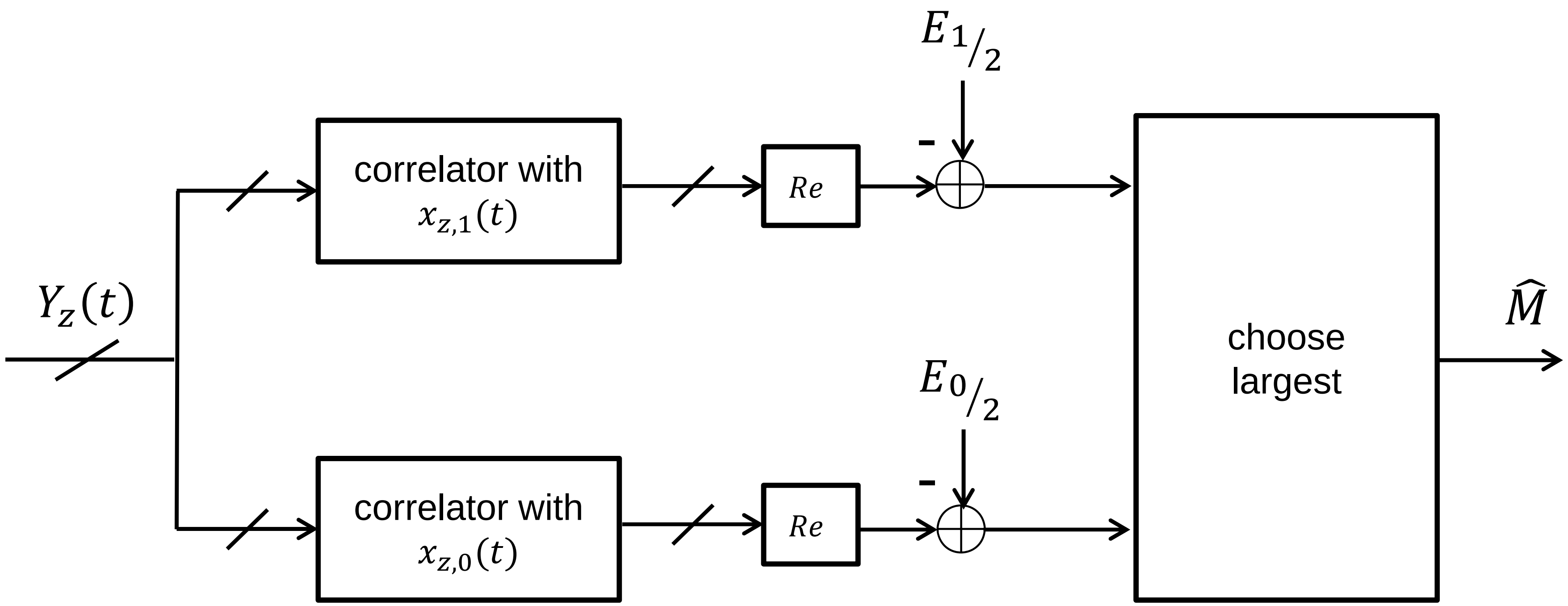
$$T_1 \underset{\widehat{M}=0}{\overset{\widehat{M}=1}{\gtrless}} T_0$$

where

$$T_i = Re\left\{\int_{-\infty}^{+\infty} Y_z(t) x_{z,i}^*(t) dt\right\} - \frac{E_i}{2}$$

likelihood metric of message $i = 0,1$

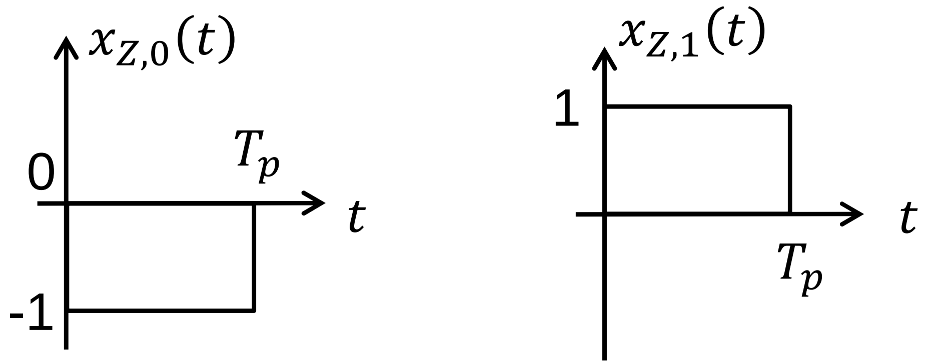
- The optimal demodulator can then also be expressed in terms of likelihood metrics as



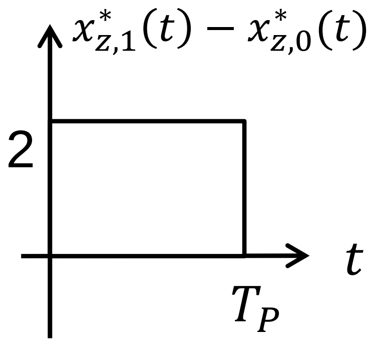
- Remark: This last implementation of the optimal filter will be convenient for the generalization to the transmission of more than one bit.

- Ex.: Given the waveforms $\{x_{z,0}(t), x_{z,1}(t)\}$ specified below, calculate the impulse response of the matched filter.

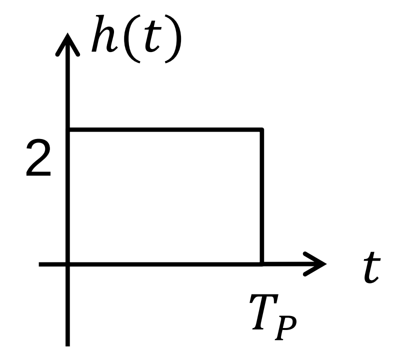
a)



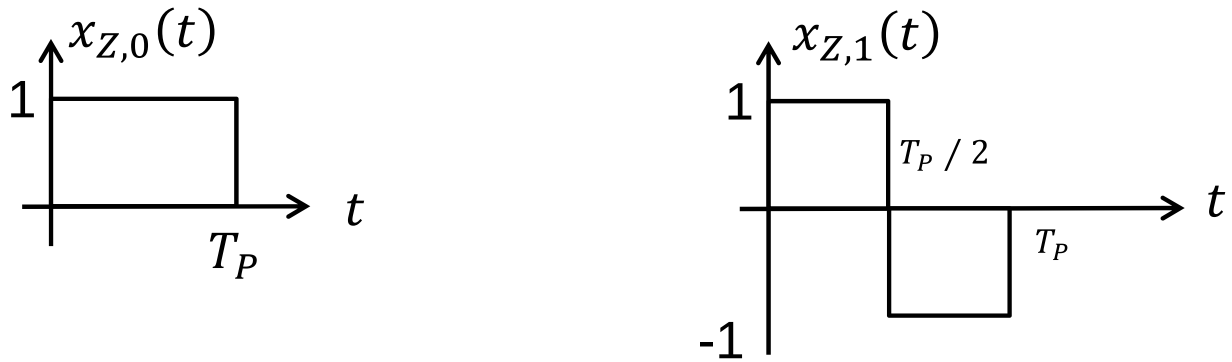
To calculate $h(t) = x^*_{z,1}(-t + T_P) - x^*_{z,0}(-t + T_P)$, let us first evaluate $x^*_{z,1}(t) - x^*_{z,0}(t)$, which is known as the effective signal:



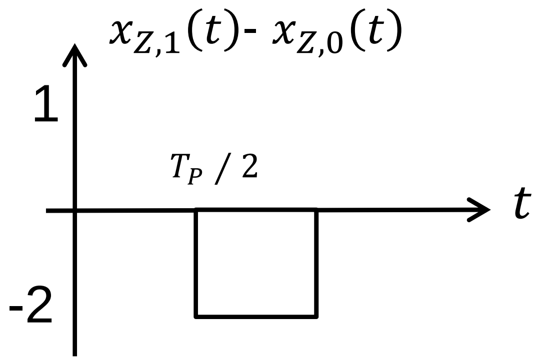
Now, by flipping around the y-axis and delaying by T_P, we get:



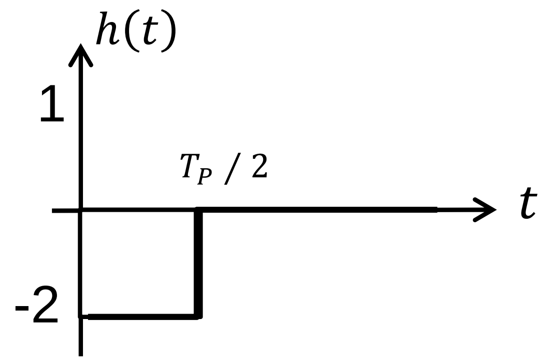
b)



Effective signal:



Matched filter:



- For additional examples, see pp. 13.20 – 13.21

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