NJIT

New Jersey's Science & Technology University

THE EDGE IN KNOWLEDGE

Optimum Waveforms

3. Optimum waveforms $\{x_{z,0}(t), x_{z,1}(t)\}$

- Given optimal γ and $H(f)$, we wish to design $\{x_{z,0}(t), x_{z,1}(t)\}$ such that the BEP

$$P_B(E) = Q(\sqrt{\eta}\,) \text{ with } \eta = \frac{\Delta_E(1,0)}{2N_0}$$

  is maximized, or equivalently the squared Euclidean distance $\Delta_E(1,0)$ is maximized

- Constraints:
  - Average energy: $E_b = \frac{E_0+E_1}{2}$
  - Bandwidth: B_T

- It is intuitive, and it can be proved as shown in the textbook, that the optimal waveforms must satisfy the condition:

$$x_{z,0}(t) = -\, x_{z,1}(t) \qquad (*)$$

- The corresponding maximum squared Euclidean distance is

$$\Delta_E(1,0) = 4E_b$$

since

$$\Delta_E(1,0) = \int_{\infty}^{+\infty} \left|x_{z,1}(t) - \left(-x_{z,1}(t)\right)\right|^2 dt = 4\int_{-\infty}^{+\infty} \left|x_{z,1}(t)\right|^2 df$$

$$= 4E_1 = 4E_0 = 4E_b$$

- The condition $(*)$ is equivalent to:

$$\rho_{01} = -1$$

and $E_0 = E_1 = E_b$

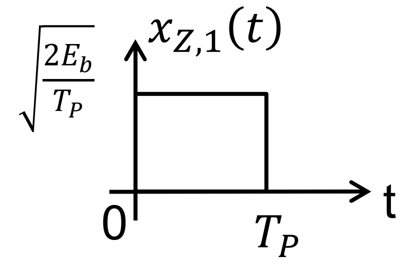
- The BEP with optimal waveforms is

$$P_B(E) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Ex.:

a) ON-OFF KEYING (OOK)

- $$\begin{cases} x_{z,0}(t) = 0 \\ x_{z,1}(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_P}} & 0 \le t \le T_P \\ 0 & elsewhere \end{cases} \end{cases}$$



- Not optimal since $x_{z,0}(t) \neq -x_{z,1}(t)$
- $E_0 = 0,\ E_1 = 2E_b \Rightarrow average\ energy = \frac{E_0+E_1}{2} = E_b$
- $\Delta_E(1,0) = E_1 = 2E_b < \underbrace{4E_b}_{\text{optimal squared Euclidean distance}}$
- $P_B(E) = \mathrm{Q}\left(\sqrt{\frac{\Delta_E(1,0)}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
- Since we need double E_b to obtain the same BEP, ON-OFF keying has a loss of 3dB $(= 10\log_{10}2)$ with respect to the optimal modulation

b) BINARY PHASE SHIFT KEYING (BPSK)

- $\begin{cases} x_{z,1}(t) = \begin{cases} \sqrt{\frac{E_b}{T_P}} & 0 \le t \le T_P \\ 0 & elsewhere \end{cases} \\ x_{z,0}(t) = -x_{z,1}(t) \end{cases}$
- Optimal since $x_{z,0}(t) = -x_{z,1}(t)$
- $E_0 = E_b, E_1 = E_b \Rightarrow average\ energy = \frac{E_0+E_1}{2} = E_b$
- $\Delta_E(1,0) = 4E_b$ optimal squared Euclidean distance
- $P_B(E) = \mathrm{Q}\left(\sqrt{\frac{\Delta_E(1,0)}{2N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{2E_b}{N_0}}\right)$

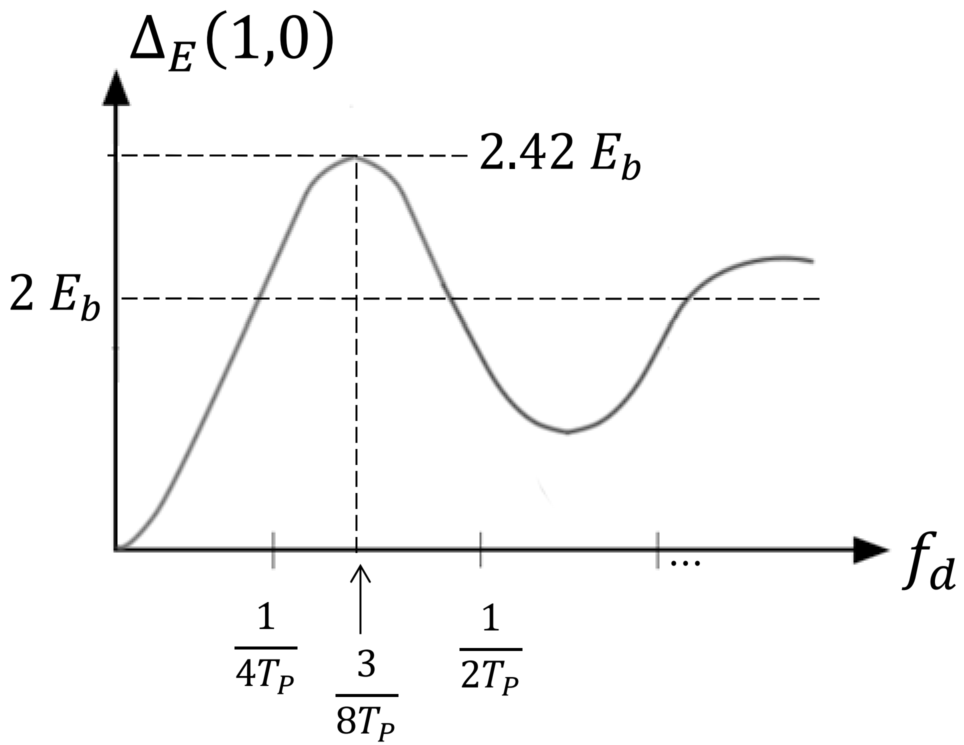
c) BINARY FREQUENCY SHIFT KEYING (BFSK)

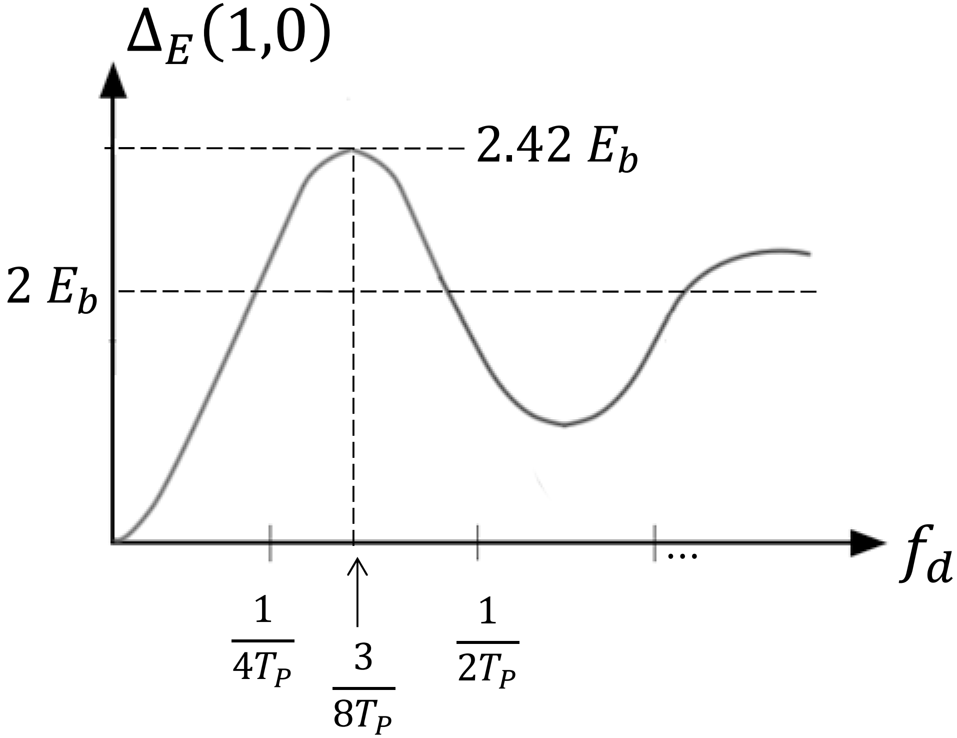
- $$\begin{cases} x_{z,0}(t) = \begin{cases} \sqrt{\frac{E_b}{T_P}} e^{j2\pi f_d t} & 0 \le t \le T_P \\ 0 & elsewhere \end{cases} \\ x_{z,1}(t) = \begin{cases} \sqrt{\frac{E_b}{T_P}} e^{-j2\pi f_d t} & 0 \le t \le T_P \\ 0 & elsewhere \end{cases} \end{cases}$$
- $f_d \ge 0$ is referred to as the frequency deviation
- The corresponding bandpass signals are

$$x_{c,0}(t) = \sqrt{2}\, Re\{x_{z,0}(t) e^{j2\pi(f_c+f_d)t}\}$$
$$= \sqrt{2}\cos(2\pi(f_c + f_d)t)$$
$$x_{c,1}(t) = \sqrt{2}\cos(2\pi(f_c - f_d)t)$$

(information encoded in the frequency)

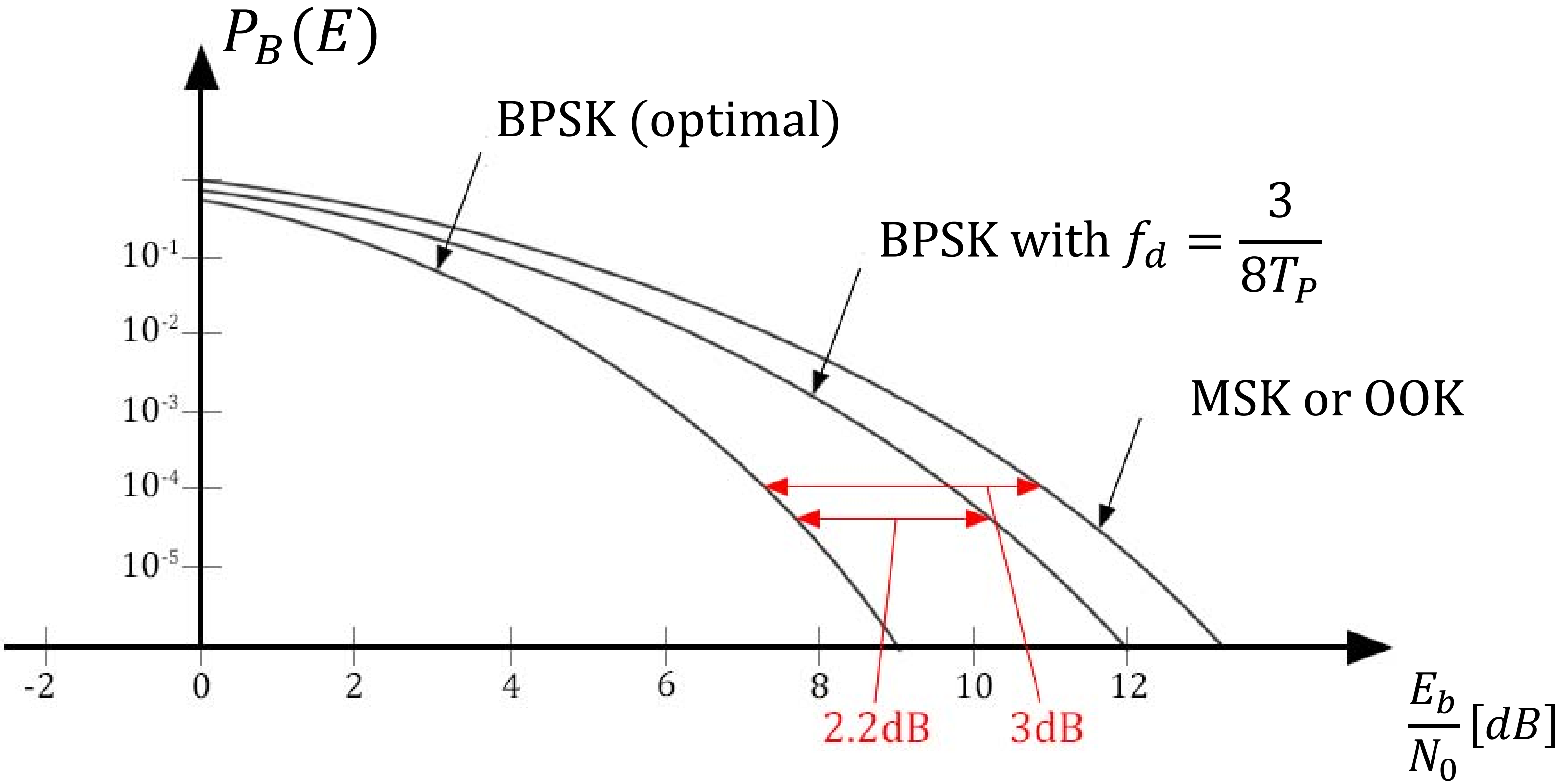
- Not optimal since $x_{z,0}(t) \neq -x_{z,1}(t)$
- $E_0 = E_b,\, E_1 = E_b \Rightarrow average\ energy = \frac{E_0+E_1}{2} = E_b$
- $\Delta_E(1,0) = 2E_b(1 - \mathrm{sinc}(4f_d T_P))$ (see textbook for calculation)
- $P_B(E) = Q\left(\sqrt{\frac{E_b}{N_0}(1 - \mathrm{sinc}(4f_d T_P))}\right)$





- The optimal f_d is $f_d = \frac{3}{8T_P}$. With this choice, we have $\Delta_E(1,0) = 2.4E_b$ and hence the loss with respect to the optimal modulation is $10\log_{10}\left(\frac{1}{0.6}\right) = 2.2dB$.
- Remark: The optimal f_d is not $f_d \to \infty$! In fact with $f_d \to \infty$, we have $\Delta_E(1,0) = 2E_b$ and hence a 3dB loss. The same result is also obtained with $f_d = \frac{1}{4T_P}$: BFSK with $f_d = \frac{1}{4T_P}$ is known as MSK. ∎

- To summarize the previous examples, the BEP is as follows



- Remark: MSK and OOK are orthogonal schemes in the sense that $\rho_{01} = 0$ (and hence $x_{z,0}(t)$ and $x_{z,1}(t)$ are orthogonal).

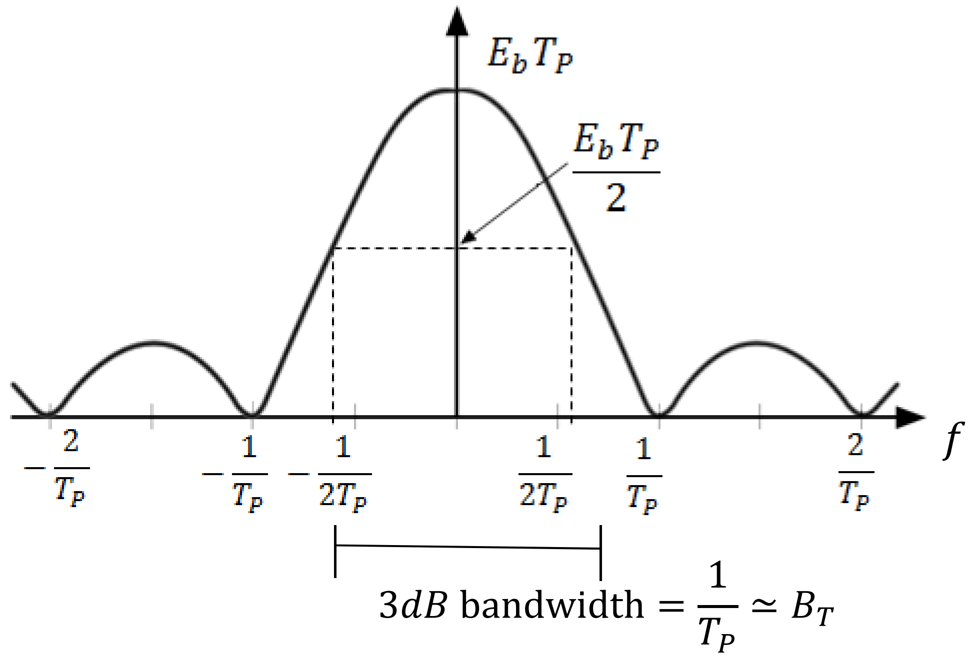
- Let us now talk about the bandwidth required by some of the mentioned modulation schemes.

Ex.:

a) BPSK

  - Average energy spectral density

$$D_{x_z}(f) = \frac{1}{2} G_{x_{z,1}}(f) + \frac{1}{2} G_{x_{z,0}}(f)$$
$$= G_{x_{z,0}}(f) = G_{x_{z,1}}(f) = \left|\mathcal{F}\{x_{z,1}(t)\}\right|^2$$
$$= E_b T_P (\mathrm{sinc}(f T_P))^2$$



- Spectral efficiency

$$\eta_B \simeq \frac{1/T_P}{1/T_P} = 1\ bits/s/Hz$$

b) BFSK

- Average energy spectral density

$$D_{x_z}(f) = \frac{1}{2} G_{x_{z,1}}(f) + \frac{1}{2} G_{x_{z,0}}(f)$$

with $G_{x_{z,0}}(f) = E_b T_P (\text{sinc}((f - f_d)T_P))^2$

$$G_{x_{z,1}}(f) = E_b T_P (\text{sinc}((f + f_d)T_P))^2$$

- It is easy to see that the bandwidth increases with f_d (see Fig. 13.23 in the textbook)
- It can be calculated that

$$B_T \simeq \begin{cases} \frac{1.5}{T_P} \text{ for } f_d = \frac{3}{8T_P} \\ \frac{1}{T_P} \text{ for MSK} \end{cases}$$

- Spectral efficiency:

$$\eta_B \simeq \begin{cases} 0.67 \quad\quad bits/s/Hz \;\; for\; f_d = \frac{3}{8T_P} \\ 1 \quad\quad bits/s/Hz \; for \text{ MSK} \end{cases}$$

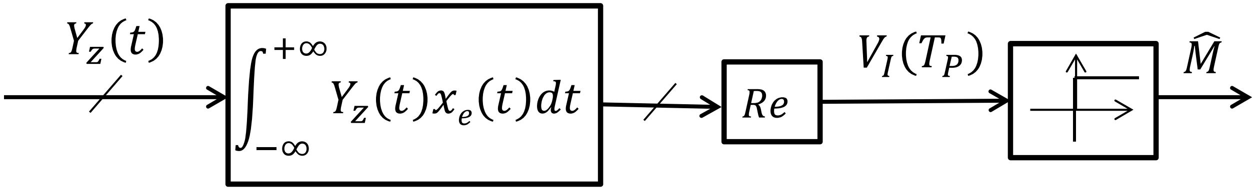
$\blacksquare$

Remark: The optimal decoder structure can often be simplified when considering specific modulation schemes.

Ex.: BPSK

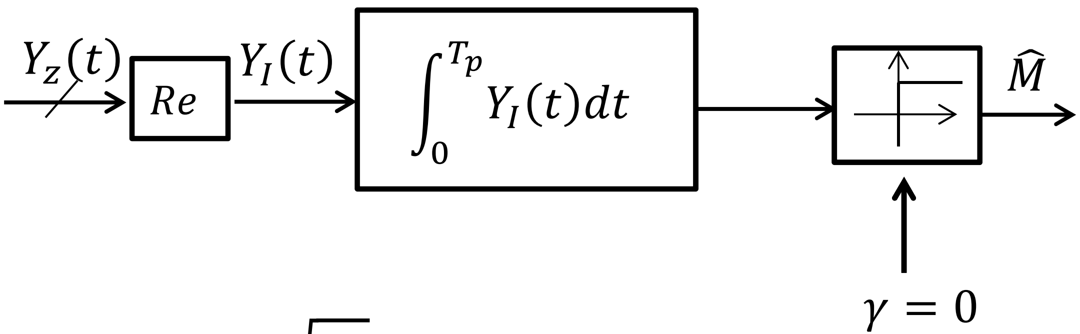
$$x_e(t) = \begin{cases} 2\sqrt{\dfrac{E_b}{T_P}} & 0 \le t \le T_P \\ 0 & elsewhere \end{cases}$$

The general structure of the optimal decoder is



(correlator
+ Re
implementation)

which can be simplified as



since $V_I(T_P) = 2\sqrt{\frac{E_b}{T_P}} \int_0^{T_P} Y_I(t)dt$

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