1. $$C = \max_{p(x)} I(X;Y) = \max_{p(m,s)} I(M,S;Y)$$

Write $p(m,s) = p(m)p(s|m)$ and define $p = \Pr[M=1]$:

$$I(M,S;Y) = \underbrace{I(M;Y)}_{\substack{= H(M) \\ = H(p)}} + \underbrace{I(S;Y|M)}_{p\,I(S;Y|M=1) + (1-p)\,I(S;Y|M=2)}$$

because $H(Y|M)=0$

$$p\,\underbrace{I(S;Y|M=1)}_{\le C_1} + (1-p)\,\underbrace{I(S;Y|M=2)}_{\le C_2}$$

with eq. iff $p(s|1)$ = capacity achieving distribution for $p(y_1|x_1)$

with eq. iff $p(s|2)$ = capacity achieving distribution for $p(y_2|x_2)$

$$\Rightarrow \quad C = \max_{0<p\le 1} \; H(p) + p\,C_1 + (1-p)\,C_2$$

The above is a convex function and the optimum can be obtained by differentiation

$$\frac{d}{dp}\left(H(p)+pC_1+(1-p)C_2\right) = \log_2 \frac{1-p}{p} + C_1 - C_2 = 0$$

$$\Rightarrow \quad 1-p = p\left(2^{C_2-C_1}\right) \Rightarrow p\left(2^{C_2-C_1}+1\right) = 1$$

$$\Rightarrow p = \frac{1}{1+\frac{2^{C_2}}{2^{C_1}}} = \frac{2^{C_1}}{2^{C_1}+2^{C_2}}$$

This leads to

$$C = \underbrace{H\left(\frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right)} + \frac{2^{C_1}}{2^{C_1}+2^{C_2}} C_1 + \frac{2^{C_2}}{2^{C_1}+2^{C_2}} C_2$$

$$- \frac{2^{C_1}}{2^{C_1}+2^{C_2}}\left(C_1 - \log\left(2^{C_1}+2^{C_2}\right)\right)$$

$$- \frac{2^{C_2}}{2^{C_1}+2^{C_2}}\left(C_2 - \log\left(2^{C_1}+2^{C_2}\right)\right)$$

$$= \log_2\left(2^{C_1}+2^{C_2}\right)$$

If $C_1 = 0$, we have $C = \log_2\left(1+2^{C_2}\right)$: the fully noisy channel can still be used to communicate information via the selection of the two channels.

2. a. $p(x,y)$:

$X \backslash Y$	0	e	1	$p(x)$
0	0.45	0.05	0	0.5
1	0	0.05	0.45	0.5
$p(y)$	0.45	0.1	0.45	

All binary sequences x^n are individually typical. Instead, we have

$$A_\epsilon^{(n)}(Y) = \{y^n \in \{0,1,e\}^n : |-\tfrac{1}{n}\log_2(0.45^{n-n_e}\, 0.1^{n_e}) - \underset{\substack{\uparrow \\ H(Y)=1.369}}{H(Y)}| \le \epsilon\}$$

$$= \{y^n \in \{0,1,e\}^n : |-\frac{n-n_e}{n}\log_2 0.45 - \frac{n_e}{n}\log_2 0.1 - 1.369| \le \epsilon\}$$

where $n_e = \sum_{i=1}^{n} 1\{Y_i = e\}$

b. In order for $(x^n, y^n) \in A_\epsilon^{(n)}(X,Y)$, we need also that

$$|-\frac{1}{n}\log_2\left((0.05)^{n_e}(0.45)^{n-n_e}\right) - \underset{\substack{\uparrow \\ H(X)+H(Y|X) \\ = 1+H(p) = 1.469}}{H(X,Y)}| \le \epsilon$$

$$\Leftrightarrow |-\frac{n_e}{n}\log_2 0.05 - \frac{n-n_e}{n}\log_2 0.45 - 1.469| \le \epsilon \quad (*)$$

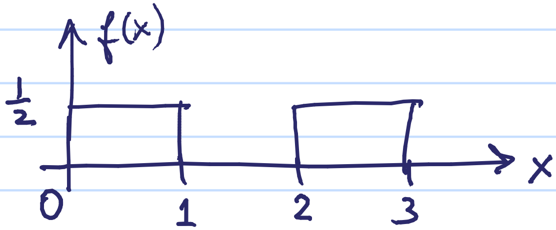
Instead $z^n \in A_\epsilon^{(n)}(Z)$ if

$$|-\frac{1}{n}\log_2\left((0.1)^{n_e}(0.9)^{n-n_e}\right) - \underset{\substack{\uparrow \\ H(p)=0.469}}{H(E)}| \le \epsilon$$

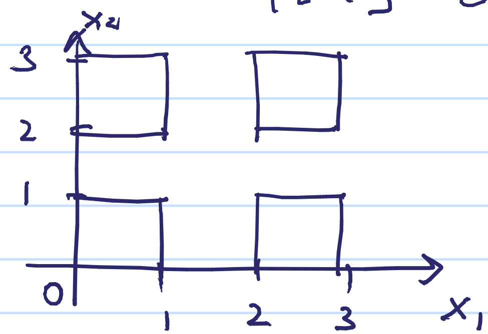
$$\Leftrightarrow |-\frac{n_e}{n}\log_2 0.1 - \frac{n-n_e}{n}\log_2 0.9 - 0.469| \le \epsilon$$

$$\Leftrightarrow (*)$$

3.



a. $A_\epsilon^{(n)} = \{ x^n \in \mathbb{R}^n : x_i \in \{[0,1] \cup [2,3]\} \text{ for all } i = 1, \dots, n \}$



$vol(A_\epsilon^{(2)}) = 4$ for any $\epsilon \geq 0$

b. From the AEP, $vol(A_\epsilon^{(n)}) \simeq 2^{nh(x)}$

where $h(X) = -2 \cdot \frac{1}{2} \int_0^1 \log_2 \frac{1}{2} \, dx = 1$

$\Rightarrow vol(A_\epsilon^{(n)}) \simeq 2^n$

which is consistent with the interpretation at part a.

4. If $Q<N$, $C=0$ since $E[Y^2] \geq E[Z^2] = N$. Otherwise:

$$I(X;Y) = h(Y) - h(Y|X)$$

$$= h(Y) - \frac{1}{2}\log_2(2\pi e N)$$

$$\leq \frac{1}{2}\log_2(2\pi e Q) - \frac{1}{2}\log_2(2\pi e N)$$

with eq. iff $X \sim N(0, Q-N)$

$$= \frac{1}{2}\log_2 \frac{Q}{N}.$$

5. $H(X,Y) \leq H(X) + H(Y)$

with equality iff $p(x,y) = p(x)p(y)$

X \ Y	0	1	2
0	1/6	1/12	1/4
1	1/12	1/24	1/8
2	1/12	1/24	1/8

6. $f(x) = e^{\lambda_0 + \lambda_1 \log g(x)} = e^{\lambda_0} (g(x))^{\lambda_1}$

where λ_0, λ_1 are such that

$$e^{\lambda_0} \int (g(x))^{\lambda_1} dx = 1 \Rightarrow e^{\lambda_0} = \frac{1}{\int (g(x))^{\lambda_1} dx}$$

and

$$e^{\lambda_0} \int (g(x))^{\lambda_1} \log g(x) dx = \alpha$$

$\Rightarrow$ if $\alpha = -h(g) = \int g(x) \log g(x) dx$, we get $\lambda_1 = 1$ and $f(x) = g(x)$.