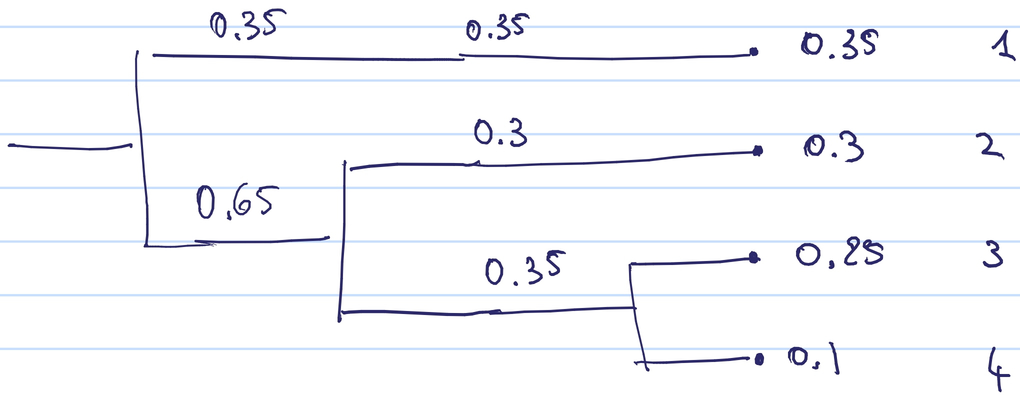
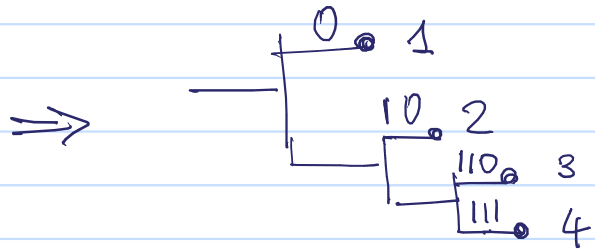
a.





X	$c(X)$	$p(X)$
1	0	0.35
2	10	0.3
3	110	0.25
4	111	0.1

b. $H(X) = -0.35 \log_2(0.35) - 0.3 \log_2(0.3) - 0.25 \log_2(0.25) - 0.1 \log_2(0.1)$

$= 1.88$

$L(C) = E[l(X)] = 0.35 \cdot 1 + 0.3 \cdot 2 + 0.25 \cdot 3 + 0.1 \cdot 3$

$= 2$

$L(C) - H(X) = 0.12$ redundancy due to the fact that the pmf is not dyadic

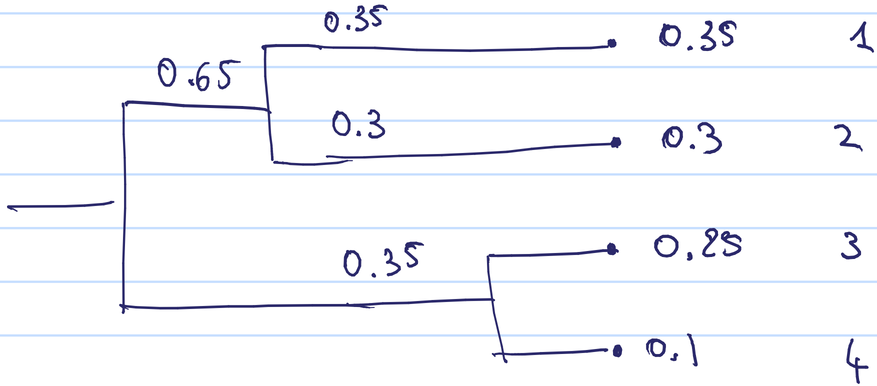
c. $\mathrm{Var}(\ell(X)) = E[\ell(X)^2] - E[\ell(X)]^2$

$E[\ell(X)^2] = 0.35 + 0.3 \cdot 4 + 0.25 \cdot 9 + 0.1 \cdot 9$

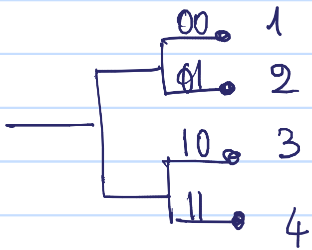
$= 4.7$

$\Rightarrow \mathrm{Var}(\ell(X)) = 4.7 - 4 = 0.7$

d.



$\Rightarrow$

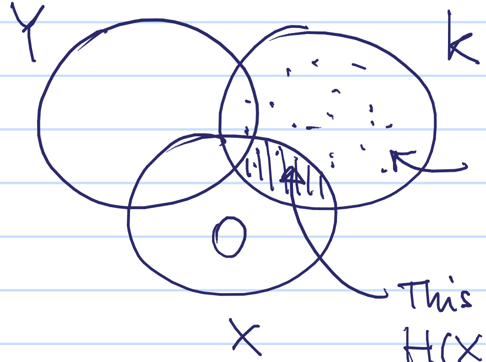


X	c(X)	p(X)
1	00	0.35
2	01	0.3
3	10	0.25
4	11	0.1

$$L(C) = E[\ell(X)] = 2$$

$$Var(X) = E[\ell(X)^2] - E[\ell(X)]^2 = 0$$

2.



We need $H(X|Y,k) = 0$

The dotted area corresponds to $H(k)$

This area corresponds to $H(X|Y)$ since $H(X|Y,k) = 0$

From the figure

$$H(K) = H(X|Y) + \underbrace{I(Y;k) + H(k|X,Y)}_{\geq 0}$$

concluding the proof.

3. $c(n) = \underbrace{1 \cdots 1}_{\substack{n-1 \\ \text{ones}}}0 \Rightarrow \ell(n) = n$ (unary coding)

$E[\ell(N)] = E[N] = H(N).$

4. a. Neyman-Pearson test:

$$\log \frac{p(X^n, Y^n)}{p(X^n)p(Y^n)} = \sum_{i=1}^{n} \log \frac{p(X_i, Y_i)}{p(X_i)p(Y_i)} \underset{H_1}{\overset{H_0}{\gtrless}} \gamma$$

threshold

b. $$\frac{1}{n} \log \frac{p(X^n, Y^n)}{p(X^n)p(Y^n)} \xrightarrow[n\to\infty]{P} E_{p(x,y)}\left[\log \frac{p(X,Y)}{p(X)p(Y)}\right] = I(X;Y)$$

c. We have

$$\Pr\left[\left|\frac{1}{n} \log \frac{p(X^n, Y^n)}{p(X^n)p(Y^n)} - I(X;Y)\right| > \epsilon\right] \to 0 \quad \text{as } n\to\infty$$

Therefore

$$\Pr\left[\frac{1}{n} \log \frac{p(X^n, Y^n)}{p(X^n)p(Y^n)} < I(X;Y) - \epsilon\right] \to 0 \quad \text{as } n\to\infty$$

so choosing

$$\gamma = n\left(I(X;Y) - \epsilon\right)$$

guarantees a vanishing probability of false alarm as $n\to\infty$.

5.

$$X \longrightarrow \boxed{f} \xrightarrow[k \text{ bits}]{W}$$

$$k \geq H(W) \geq H(X)$$