ECE 232 - Circuits and Systems II
Final

Please provide clear and complete answers. Don't forget to specify the units of measure!

Q1. (2 point) Calculate and sketch the convolution $v_O(t) = v_I(t) * h(t)$ between the two rectangles shown in Fig. 1.

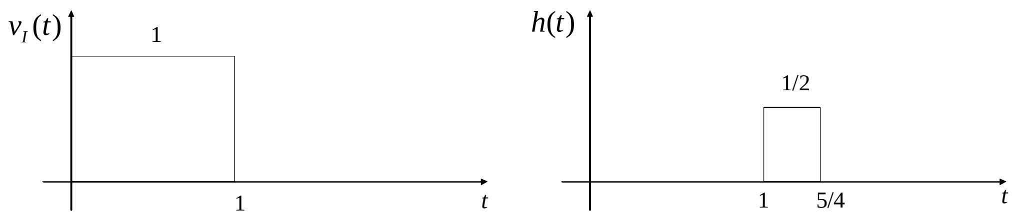

Figure 1:

Sol.: Using the time-invariant property of convolution, it is easy to calculate

$$\begin{aligned} v_O(t) &= 0 \text{ for } t \leq 1 \\ v_O(t) &= \frac{1}{2}(t-1) \text{ for } 1 \leq t \leq \frac{5}{4} \\ v_O(t) &= \frac{1}{8} \text{ for } \frac{5}{4} \leq t \leq 2 \\ v_O(t) &= \frac{1}{2}\left(-t+\frac{9}{4}\right) \text{ for } 2 \leq t \leq \frac{9}{4} \\ v_O(t) &= 0 \text{ for } t > \frac{9}{4}. \end{aligned}$$

Q2. (2 point) Assume that the input to a circuit with transfer function $H(s) = \frac{1}{s+3}$ is given by a function $v_I(t)$ that is periodic with period $T = 1$ second and has Fourier series coefficients $a_0 = \frac{1}{2}$, $a_k = 0$ for $k = 1, 2, ...$, $b_1 = -1$ and $b_k = 0$ for $k = 2, 3, ...$ Find the steady-state output $v_O(t)$.

Sol.: Since the fundamental frequency is $\omega_0 = \frac{2\pi}{T} = 2\pi$, the input is:

$$v_I(t) = \frac{1}{2} - \sin(2\pi t).$$

Defining the transfer function as $H(j\omega) = |H(j\omega)|e^{j\theta(j\omega)}$, it follows from the superposition principle that the output is

$$\begin{aligned} v_O(t) &= \frac{1}{2}H(j0) - |H(j2\pi)|\sin(2\pi t + \theta(j2\pi)) \\ &= \frac{1}{2} - 0.14\sin(2\pi t - 64.4^\circ) \end{aligned}$$

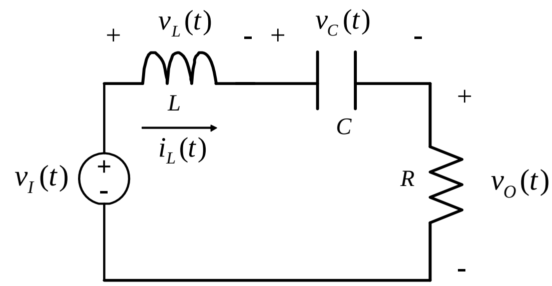

Figure 2:

since $H(j2\pi) = \frac{1}{j2\pi+3} = 0.14e^{-j64.4°}$.

P1 (4 points). Consider the series RLC circuit in Fig. 2.
1.a. What type of filter is it? Justify your response.
1.b. We want to design such filter so that the quality factor is $Q = 2$, the central frequency is $10kHz$ and $C = 1\mu F$. Find R and L.
1.c. Find the transfer function of the filter, calculate poles and zeros, and cut-off frequencies (check that the bandwidth is as required). Sketch the frequency response.
1.d. Assume that the input is $v_I(t) = \cos(2\pi \cdot 10000t)u(t)$ and that the energy stored at time $t = 0^-$ is zero on both capacitor and inductor. Find the Laplace transform of the output $v_O(t)$. What are the corresponding steady-state and transient poles? Find the steady-state output.

Sol.:
1.a. It is a bandpass filter as it can be easily seen by checking the behavior for $\omega = 0$, $\omega = \infty$ and showing that for $\omega_0 = 1/\sqrt{LC}$ we have $V_O(s) = V_I(s)$ (see class notes and text).
1.b. We have

$$\begin{aligned} \omega_0 &= 2\pi \times 10^4 = \frac{1}{\sqrt{LC}} \\ &\rightarrow L = \frac{1}{C}\frac{1}{(2\pi \times 10^4)^2} = \frac{10^{-2}}{(2\pi)^2} \\ &\simeq 2.5 \times 10^{-4} \text{ F} \end{aligned}$$

Moreover, we have

$$Q = \frac{\omega_0}{2\alpha} = 2,$$

where $\alpha = \frac{R}{2L}$, so that

$$\begin{aligned} Q &= \frac{\omega_0 L}{R} = 2 \\ &\rightarrow R = \frac{\omega_0 L}{2} = \frac{2\pi \times 10^4 \times \frac{10^{-2}}{(2\pi)^2}}{2} \\ &= \frac{10^2}{4\pi} \simeq 8 \ \Omega. \end{aligned}$$

Notice that $\alpha = \pi/2 \times 10^4 = 1.57 \times 10^4$.
1.c. The transfer function is given by

$$\begin{aligned} H(s) &= \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2} \\ &= \frac{\pi \times 10^4 s}{s^2 + \pi \times 10^4 s + (2\pi)^2 \times 10^8}. \end{aligned}$$

There is one zero in $s = 0$. Since $\omega_0 > \alpha$, poles are given by

$$\begin{aligned} s_1 &= -\alpha + j\sqrt{\alpha^2 - \omega_0^2} = (-\frac{1}{2} + j\sqrt{\frac{15}{4}})\pi \times 10^4 \\ &\simeq (-1.57 + j6.08) \times 10^4 \qquad (1a) \\ s_2 &= -\alpha - j\sqrt{\alpha^2 - \omega_0^2} = (-\frac{1}{2} - j\sqrt{\frac{15}{4}})\pi \times 10^4 \\ &\simeq (-1.57 - j6.08) \times 10^4. \qquad (1b) \end{aligned}$$

The cut-off frequencies are

$$\begin{aligned} \omega_{c1} &= -\alpha + \sqrt{\omega_0^2 + \alpha^2} \simeq 49\ krad/s \\ \omega_{c2} &= \alpha + \sqrt{\omega_0^2 + \alpha^2} \simeq 80\ krad/s. \end{aligned}$$

For a sketch of the frequency response, see the textbook.
1.d. The Laplace transform of the output is

$$\begin{aligned} V_O(s) &= H(s)V_I(s) = \\ &= \frac{\pi \times 10^4 s}{s^2 + \pi \times 10^4 s + (2\pi)^2 \times 10^8} \cdot \frac{s}{s + (2\pi)^2 \times 10^8}. \end{aligned}$$

The steady-state poles are $\pm j2\pi \times 10^4$, whereas the transient poles are $(-1.57 \pm j6.08) \times 10^4$. The steady-state output is given by

$$\begin{aligned} v_O(t) &= |H(j2\pi \times 10^4)| \cos(2\pi \times 10^4 + \theta(j2\pi \times 10^4)) \\ &= \cos(2\pi \times 10^4 + \theta(j2\pi \times 10^4)) \end{aligned}$$

since $|H(j2\pi \times 10^4)| = 1$ and $\theta(j2\pi \times 10^4) = 0$ given that the input $v_I(t)$ has frequency equal to the central frequency.

P2 (4 points). For the same circuit at the previous point, assume that the initial conditions are $v_C(0^-) = 1V$ and $i_L(0^-) = 0\ A$. Moroever, the voltage source is off (i.e., equal to zero) for all times $t \geq 0$. (If you were not able to solve the previous point, set $L = 0.25\ mH$, $R = 8$ Ω and $C = 1\ \mu F$)
2.a. Find the output $v_O(t)$ *by working in the time domain only* (do not use the Laplace transform!).
2.b. Add a resistor with equal resistance R in parallel to the resistor in the circuit of Fig. 2. Does $v_O(t)$ decay slower or faster than at the previous point 2.a? Quantify by finding the time constants for the two cases.

Sol.:
2.a. As seen at the previous point, we have $\omega_0 > \alpha$, hence the circuit operates in the underdamped regime. Since the source is off, we are interested in evaluating the natural response. We can then use the general formula for $v_O(t)$, which is given by:

$$\begin{aligned} v_O(t) &= e^{-\alpha t}(B_1\cos(\omega_d t) + B_2\sin(\omega_d t)) \\ &= e^{-1.57\times 10^4 t}(B_1\cos(6.08\times 10^4 t) + B_2\sin(6.08\times 10^4 t)), \end{aligned}$$

where α and ω_d follow from (1). Imposing the initial conditions, we get

$$v_O(0) = B_1 = 0,$$

since the initial current in the inductor is zero. Moreover,

$$\begin{aligned} v_O(t) &= Ri_L(t) \\ &\rightarrow \frac{dv_O(t)}{dt} = R\frac{di_L(t)}{dt} = \frac{R}{L}v_L(t) \\ &= \frac{R}{L}(-v_O(t) - v_C(t)) \end{aligned}$$

and

$$\frac{dv_O(0)}{dt} = -\alpha B_1 + \omega_d B_2 = \omega_d B_2,$$

so that

$$\begin{aligned} -\alpha + \omega_d B_2 &= \frac{R}{L}(-v_O(0) - v_C(0)) \\ &= \frac{R}{L}(-v_C(0)) \\ &= \frac{\frac{10^2}{4\pi}}{\frac{10^{-2}}{(2\pi)^2}}(-1) \\ &= -\pi\times 10^4 \\ &\rightarrow B_2 = \frac{-\pi\times 10^4}{6.08\times 10^4} = -0.52. \end{aligned}$$

We thus have

$$v_O(t) = -0.52e^{-1.57\times 10^4 t}\sin(6.08\times 10^4 t).$$

Notice that the time constant is $\frac{1}{1.57\times 10^4} = 0.63\times 10^{-4}$ sec.
2.b. The equivalent resistance becomes $R_{eq} = R/2 = \frac{10^2}{8\pi} \simeq 4\ \Omega$. Therefore, α is decreased to

$$\alpha = \frac{R_{eq}}{2L} = 0.78\times 10^4,$$

so that the time constant is doubled to 1.27×10^{-4} sec.