ECE 232 - Circuits and Systems II
Midterm

Please provide clear and complete answers. Don't forget to specify the units of measure!

Q1. (1 point) Consider the parallel of a resistor with $R = 1k\Omega$ and a capacitor with $C = 1\mu F$, and denote as $v(t)$ the voltage at the common terminals of resistor and capacitor. Can we have $v(t) = \sin(10^{-3}t)$ $[V]$ for $t \geq 0$? Justify your answer.

Sol.: No, since the current in the resistor, $i_R(t)$, and capacitor, $i_C(t)$, must satisfy the Kirchoff current law:

$$i_R(t) = -i_C(t),$$

but also the relationships for resistor and capacitor:

$$\begin{aligned} i_R(t) &= \frac{v(t)}{R} = 10^{-3}\sin(10^{-3}t)\ A \\ \text{and } i_C(t) &= C\frac{dv}{dt} = 10^{-9}\cos(10^{-3}t)\ A. \end{aligned}$$

These conditions are incompatible!

Q2. (1 point) Calculate the Laplace transform of

$$f(t) = e^{-10(t-0.03)}\cos(100t-3)u(t-0.03),$$

starting from the Laplace transform $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$ and using the properties of the Laplace transform. (Hint: Recall the translation in time and frequency properties...)

Sol.: We can write

$$\begin{aligned} f(t) &= e^{-10(t-0.03)}\cos\left(100\left(t-\frac{3}{100}\right)\right)u(t-0.03) \\ &= e^{-10(t-0.03)}\cos\left(100\left(t-0.03\right)\right)u(t-0.03). \end{aligned}$$

Using the translation in the frequency domain property, we get

$$\mathcal{L}\{e^{-10t}\cos(100t)u(t)\} = \frac{s+10}{(s+10)^2+10^4},$$

then, using the translation in time domain property, we get

$$\begin{aligned} &\mathcal{L}\{e^{-10(t-0.03)}\cos\left(100\left(t-0.03\right)\right)u(t-0.03)\} \\ = &\frac{s+10}{(s+10)^2+10^4}e^{-0.03s}. \end{aligned}$$

P1. (4 points) Consider the circuit in figure 1. It is known that $v_C(0^-) = 10\ V$. Moreover, the switch has been in position a for a long time before moving to position b at time $t = 2$ ms.

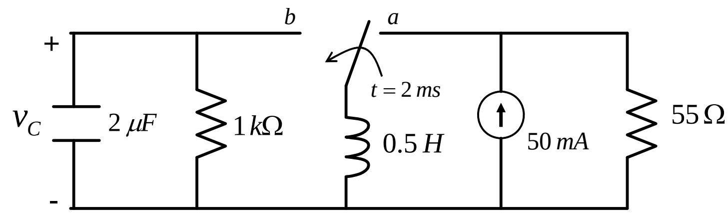

Figure 1:

1.a) Calculate $v_C(t)$ for $0 \leq t \leq 2\ ms$ (i.e., before the switch moves to position b).
1.b) Calculate $v_C(t)$ for $t \geq 2\ ms$.

Sol.: 1.a) For $0 \leq t \leq 2\ ms$, we can concentrate on the RC circuit on the left, with $R = 1$ $k\Omega$ and $C = 2\ \mu F$. This has time constant

$$\tau = RC = 2\ ms,$$

so that we have

$$v_C(t) = v_C(0^-)e^{-t/\tau} = 10e^{-500t}$$

for $0 \leq t \leq 2\ ms$.
1.b) For $t \geq 2\ ms$ we can concentrate on the RLC circuit on the left with $R = 1\ k\Omega$, $C = 2$ μF and $L = 0.5\ H$, which is characterized by

$$\begin{aligned} \alpha &= \frac{1}{2RC} = 250 \text{ rad/s} \\ \omega_0 &= \frac{1}{\sqrt{LC}} = 1000 \text{ rad/s} \end{aligned}$$

Moreover, the initial conditions are $v_C(2 \times 10^{-3}) = 10e^{-1} = 3.67\ V$ and $i_L(2 \times 10^{-3}) = 50$ mA. The latter fact follows from the analysis of the right part of the circuit for $t < 2\ ms$. The circuit operates in the underdamped regime ($\omega_0 > \alpha$) and the solution is given by

$$\begin{aligned} v_C(t) &= B_1 e^{-250(t-2\times10^{-3})} \cos(968.2(t - 2 \times 10^{-3})) \\ &\quad + B_2 e^{-250(t-2\times10^{-3})} \sin(968.2(t - 2 \times 10^{-3})) \end{aligned}$$

with damped frequency given by

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 968.2 \text{ rad/s}.$$

Imposing the initial conditions, we get:

$$\begin{aligned} B_1 &= 3.67 \\ -250B_1 + 968.2B_2 &= -\frac{1}{C}\left(\frac{v_C(2 \times 10^{-3})}{R} + i_L(2 \times 10^{-3})\right) \\ &= -\frac{1}{2 \times 10^{-6}}\left(3.67 \times 10^{-3} + 50 \times 10^{-3}\right) \\ &\rightarrow B_2 = \frac{1}{968.2}\left(250 \times 3.67 - 26.8 \times 10^3\right) \\ &= -26.7, \end{aligned}$$

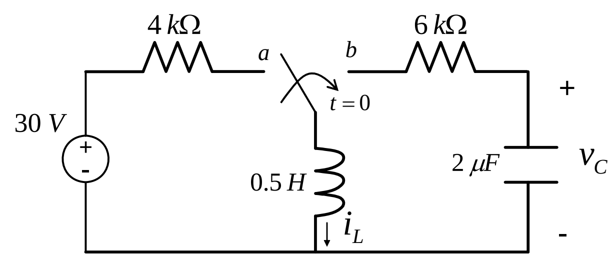

Figure 2:

so that

$$\begin{aligned} v_C(t) &= 3.67e^{-250(t-2\times10^{-3})}\cos(968.2(t-2\times10^{-3})) \\ &\quad -26.7e^{-250(t-2\times10^{-3})}\sin(968.2(t-2\times10^{-3})). \end{aligned}$$

P2. (4 points) Consider the circuit in figure 2. The switch has been in position a for a long time before it moves to position b at time $t = 0$. The energy stored in the capacitor at time $t = 0^-$ is zero.
2.a) Find initial current $i_L(0^-)$ and construct an equivalent circuit in the s-domain for $t > 0$ (use whatever representation you find more convenient for both inductor and capacitor in the s-domain).
2.b) Find $I_L(s)$ and $V_C(s)$ (Express all coefficients in numerical form).
2.c) Calculate poles and zeros of $I_L(s)$ and $V_C(s)$.
2.d) Now substitute the capacitor with a short circuit. Everything else remains the same. Find $I_L(s)$ *and* the time-domain current $i_L(t)$ for $t \geq 0$. What is the time constant of the circuit?

Sol.:
2.a) We have

$$i_L(0^-) = \frac{30}{4\times10^3} = 7.5\ mA$$

and

$$v_C(0^-) = 0\ V,$$

since the circuit is in steady-state for $t = 0^-$.
An equivalent circuit in the s-domain is shown in figure 3 using the series equivalent for the inductor (the capacitor has zero initial conditions). Notice that the voltage source is $Li_L(0^-) = 0.5 \times 7.5 \times 10^{-3} = 3.75 \times 10^{-3}$ $[V \times \text{sec}]$.
2.b) Using the representation above for the circuit, we find

$$\begin{aligned} I_L(s) &= \frac{3.75\times10^{-3}}{0.5s + 6\times10^3 + \frac{1}{2s}10^6} \\ &= \frac{7.5\times10^{-3}s}{s^2 + 12\times10^3 s + 10^6} \end{aligned}$$

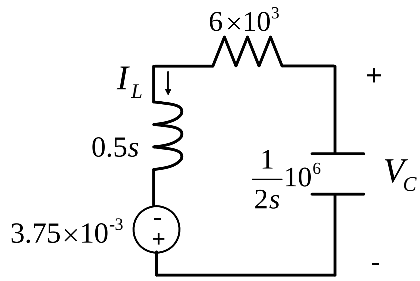

Figure 3:

and

$$
\begin{aligned}
V_C(s) &= -\frac{1}{2s}10^6 \times I_L(s) \\
&= -\frac{3.75 \times 10^3}{s^2 + 12 \times 10^3 s + 10^6}.
\end{aligned}
$$

2.c) $I_{:L}(s)$ has a zero in $s = 0$, unlike $V_C(s)$ that has no zeros. The poles are the same for both $I_L(s)$ and $V_C(s)$, and given by

$$
\begin{aligned}
s_1 &= -83.9 \\
s_2 &= -11,916
\end{aligned}
$$

Notice that these are the roots of the characteristic polynomial of the RLC circuit at hand, namely

$$
\begin{aligned}
s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -83.9 \\
s_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -11,916
\end{aligned}
$$

where $\alpha = \frac{R}{2L} = 6 \times 10^3$ and $\omega_0 = \frac{1}{\sqrt{LC}} = 1000$.
d) If there is no capacitor, the circuit for $t \geq 0$ is RL and we have from figure 3

$$
\begin{aligned}
I_L(s) &= \frac{3.75 \times 10^{-3}}{0.5s + 6 \times 10^3} = \\
&= \frac{7.5 \times 10^{-3}}{s + 12 \times 10^3},
\end{aligned}
$$

which leads to

$$
i_L(t) = 7.5e^{-12000t} \text{ for } t \geq 0.
$$

The time constant of the circuit is

$$
\tau = \frac{L}{R} = \frac{1}{12000} = 0.08\ ms.
$$