ECE 232 - Circuits and Systems II
Test 3

Please provide clear and complete answers. Don't forget to specify the units of measure!

1. Consider the circuit in Fig. 1-(a).
1.a. Find the impulse response $h(t)$.
1.b. Find the output voltage $v_O(t)$ when the input is $v_I(t)$ as shown in Fig. 1(b) (and the capacitor has zero energy at time $t = 0^-$). To do so, evaluate the convolution integral $v_O(t) = v_I(t) * h(t)$.
1.c. Sketch the result.

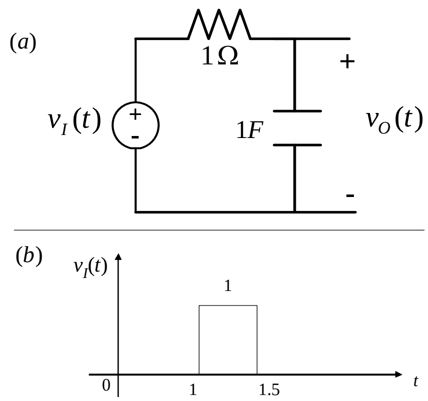

Figure 1:

Sol.: 1.a. The transfer function is

$$H(s) = \frac{1/s}{1+1/s} = \frac{1}{s+1},$$

so that the impulse response is

$$h(t) = e^{-t}u(t).$$

1.b. The input $v_I(t)$ can be written as

$$v_I(t) = rect(t-1),$$

where $rect(t)$ is a rectangle between 0 and 0.5 with height 1. In other words, $v_I(t)$ is a regular rectangle of duration 0.5 delayed by 1. The output $v_O(t)$ can thus be obtained by calculating the convolution

$$rect(t) * h(t)$$

and then delaying the result by 1 by time-invariance [1].
The convolution $rect(t) * h(t)$ is given by:

[1] You can also evaluate directly the convolution $v_I(t) * h(t)$ but this approach is easier.

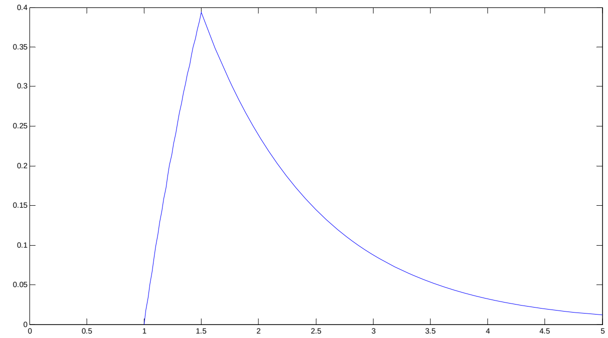

Figure 2:

- If $t \leq 0$, $rect(t) * h(t) = 0$;
- If $0 \leq t \leq 0.5$
$$rect(t) * h(t) = \int_0^t e^{-\tau} d\tau = 1 - e^{-t};$$
- If $t \geq 0.5$
$$\begin{aligned} rect(t) * h(t) &= \int_{t-0.5}^t e^{-\tau} d\tau = e^{-(t-0.5)} - e^{-t} \\ &= e^{-t}(e^{0.5} - 1). \end{aligned}$$

We thus obtain $v_O(t)$ as

- If $t \leq 1$
$$v_O(t) = 0;$$
- If $1 \leq t \leq 1.5$
$$v_O(t) = 1 - e^{-(t-1)};$$
- If $t \geq 1.5$
$$v_O(t) = e^{-(t-1)}(e^{0.5} - 1).$$

The result is shown in the figure below.

2. Add a resistor with resistance equal to 1Ω in parallel to the capacitor in the circuit of Fig. 1-(a).
2.a. What type of filter do you obtain? What is the cut-off frequency and the maximum amplitude response of this filter?
2.b. (Extra) Calculate and plot the amplitude and phase responses.

Sol.: 2.a. The transfer function is

$$\begin{aligned}
H(s) &= \frac{(1||1/s)}{1+(1||1/s)} \\
&= \frac{\frac{1/s}{1+1/s}}{1+\frac{1/s}{1+1/s}} \\
&= \frac{1}{s+2}.
\end{aligned}$$

This is a low-pass filter with cut-off frequency $\omega_c = 2$. In fact, it can be written as

$$H(s) = \frac{1}{2} \cdot \frac{2}{s+2},$$

where $\frac{2}{s+2}$ is the standard transfer function of an RC low-pass filter with cut-off frequency $\omega_c = 2$. The maximum amplitude response of the low-pass filter is obtained for zero frequency and is given by $1/2$. Notice that the presence of the load resistor has modified both the cut-off frequency and the maximum amplitude response as compared to the original filter in Fig. 1-(a).

2.b. We obtain

$$\begin{aligned}
|H(j\omega)| &= \frac{1}{\sqrt{4+\omega^2}} \\
\theta(j\omega) &= -\angle(2+j\omega).
\end{aligned}$$