ECE 776 - Midterm Spring 2014

Please provide detailed answers.

1. (2 points) Consider random variables $X \backsim Ber(0.1)$ and $Y = X + Z$ where $Z \backsim Ber(0.3)$ is independent of X.

a. Find the best estimator of X given Y and its probability of error.

b. Compare the result at the previous point with Fano's inequality.

2. (1 point) Consider a stationary process $X_1, X_2, ...$ Prove that $H(X_i|X_1, ..., X_{i-1})$ is non-increasing with i.

3. (3 points) We are given two pmfs $p(x) = (1/12, 1/2, 1/4, 1/6)$ and $q(x) = (0, 1/2, 1/4, 1/4)$.

a. Assume that $q(x)$ is the correct pmf of a memoryless source, but a Huffman code C is constructed using the wrong pmf $p(x)$. What is the redundancy $L(C) - H(X)$ of this code?

b. Compare the result above with the redundancy of a Shannon code constructed using the wrong pmf $p(x)$. Connect this result to the KL divergence between $p(x)$ and $q(x)$.

c. Assume now that $p(x)$ is the correct pmf of a memoryless source, but a Huffman code is constructed using the wrong pmf $q(x)$. What can you say about the resulting code?

4. (2 points) We want to generate a random variable $Y \sim p(x) = (1/2, 1/4, 1/4)$. To this end, we have available a fair coin that we can toss independently multiple times, i.e., an iid sequence of variables $X_i \sim Ber(0.5)$. Find a way to generate Y from multiple tosses of the coin. Show that this scheme requires on average a number of coin tosses equal to $H(Y)$.

5. (1 point) Show that an arithmetic code, which has $l(x) = \lceil -\log_2 p(x) \rceil + 1$ and $c(x) = \lfloor \bar{F}(x) \rfloor_{l(x)}$, is prefix free.

6. (2 points) A $Ber(p)$ memoryless source is compressed using a fixed-to-variable Shannon code that operates over blocks of size k. Note that the per-symbol length $l(X^k)/k$ of the resulting codeword is a random variable. You can approximate $l(X^k)$ with the ideal codeword length in order to simplify the problem.

a. As k grows, what happens to $l(X^k)/k$?

b. We wish to have a variance of $l(X^k)/k$ smaller than 0.1. How large should k be if $p = 0.1$?