ECE 776 - Information theory
Final

Q1. (1 point) We would like to compress a Gaussian source with zero mean and variance 1. We consider two strategies. In the first, we quantize with a step size Δ so that the distortion $D = \Delta^2/12$ is equal to 0.1 and then use entropy encoding on the quantized samples X_i^Δ. In the second strategy, we use optimal rate-distortion coding with distortion $D = 0.1$. Compare the number of bits per symbol needed for the two strategies (in particular, for the first use results valid in the limit of a small Δ).

Sol.: For the first strategy, we have $D = \Delta^2/12 = 0.1$ so that $\Delta = \sqrt{1.2} = 1.1$

$$R = H(X^\Delta) = -\log_2 \Delta + h(X) = -\log_2(1.1) + \frac{1}{2}\log_2(2\pi e) = 1.91 \text{ bits/symbol},$$

while for the second

$$R(0.1) = \frac{1}{2}\log_2\left(\frac{1}{0.1}\right) = 1.66 \text{ bits/symbol}.$$

Clearly, the optimal code according to rate-distortion theory requires a smaller number of bits/symbol.

Q2. (1 point) Find the rate-distortion function $R(D)$ for a Bernoulli source $X \sim Ber(0.6)$ with distortion

$$d(x,\hat{x}) = \begin{cases} 0 & x = \hat{x} \\ \infty & x = 1, \hat{x} = 0 \\ 1 & x = 0, \hat{x} = 1 \end{cases}.$$

Sol.: In order to obtain an average distortion D, we should have

$$D = \sum_{x,\hat{x}} p(x,\hat{x})d(x,\hat{x}).$$

It is easy to see that there exists only one joint pmf $p(x,\hat{x})$ that satisfies the above conditions and the constraint on the marginal pmf of X stated in the text:

$X/\hat{X}$	0	1
0	$0.4 - D$	D
1	0	0.6

To see this, notice that with this choice the following conditions are satisfied:

$$\begin{aligned} D &= \sum_{x,\hat{x}} p(x,\hat{x})d(x,\hat{x}) = p(0,1) \\ p_X(1) &= 0.6. \end{aligned}$$

The rate-distortion function can be then obtained for $D < 0.4$ as

$$\begin{aligned} R(D) &= I(X;\hat{X}) = H(X) - H(X|\hat{X}) = H(0.6) - p_{\hat{X}}(1)H(X|\hat{X}=1) = \\ &= H(0.6) - (D+0.6)H(D/(D+0.6)). \end{aligned}$$

Notice that for $D > 0.4$, $R(D) = 0$.

Q3. (1 point) Assume that you have two parallel Gaussian channels with inputs X_1 and X_2 and noises Z_1 and Z_2, respectively. Assume that the noise powers are $E[Z_1^2] = 0.3$ and $E[Z_2^2] = 0.6$, while the total available power is $P = E[X_1^2] + E[X_2^2] = 0.1$. Find the optimal power allocation (waterfilling) and corresponding capacity.

Sol.: The waterfilling conditions are:

$$\begin{aligned} P_1 &= E[X_1^2] = (\mu - 0.3)^+ \\ P_2 &= E[X_2^2] = (\mu - 0.6)^+ \end{aligned}$$

with

$$P_1 + P_2 = 0.1.$$

It follows that

$$(\mu - 0.3)^+ + (\mu - 0.6)^+ = 0.1.$$

If you assume both P_1 and $P_2 > 0$, it is easy to see that there is no solution to the previous equation. Instead, if we set $P_2 = 0$, we get $(\mu - 0.3) = 0.1$, from which we obtain $\mu = 0.4$ and

$$\begin{aligned} P_1 &= (0.4 - 0.3)^+ = 0.1 \\ P_2 &= (0.4 - 0.6)^+ = 0, \end{aligned}$$

i.e., all the power is used on the best channel. The corresponding capacity is then

$$C = \frac{1}{2} \log_2 \left(1 + \frac{0.1}{0.3} \right) = 0.21 \text{ bits/symb.}$$

Q4. (1 point) Two sensors collect information about the temperature of a given process in different locations. At each time instant, they both need to communicate whether the temperature of the process is above a given threshold (alarm) or not (normal). Since the sensors are in adjacent areas, their measurements U and V are correlated according to the joint PMF

$U \backslash V$	normal	alarm
normal	0.8	0.05
alarm	0.05	0.1

How many bits R_1 and R_2 are needed to deliver this information if U and V are encoded independently? What if we use Slepian-Wolf source coding?

Sol.: If U and V are decoded independently we need

$$\begin{aligned} R_1 &\geq H(U) = H(0.85) = 0.61 \\ R_2 &\geq H(V) = 0.61, \end{aligned}$$

and a total number of bits $R_1 + R_2 \geq 0.61 + 0.61 = 1.22$. On the contrary, if we use Slepian-Wolf coding, we only need

$$\begin{aligned} R_1 &\geq H(U|V) = 0.85 \cdot H\left(\frac{0.8}{0.85}\right) + 0.15 \cdot H\left(\frac{0.05}{0.15}\right) = 0.41 \\ R_2 &\geq H(V|U) = 0.41 \\ R_1 + R_2 &\geq H(U,V) = -2 \cdot 0.05 \cdot \log_2(0.05) - 0.1 \cdot \log_2(0.1) - 0.8 \cdot \log_2(0.8) = 1.02 \end{aligned}$$

so that the total number of bits needed is only $R_1 + R_2 \geq 1.02$ provided that $R_1 \geq 0.32$ and $R_2 \geq 0.32$.

P1. (2 points) We would like to transmit a Bernoulli source $X_i \sim Ber(0.7)$ with Hamming distortion D over a binary symmetric channel (BSC) with probability of error p.
(a) Assuming that we send one source symbol for each channel symbol, can the source X_i be transmitted losslessly ($D = 0$) over the BSC if $p = 0.3$?

Sol.: The condition to be verified is that $R(0) = H(X) \leq C = 1 - H(p)$. However, since $H(X) = H(0.7) = 0.88$ bits/ source symbol and $C = 1 - H(0.3) = 0.12$ bits/ channel symbol, the condition does not hold true, and, as a result, the source cannot be transmitted over the channel.

(b) In the same condition as at point (a), can we send source X_i with distortion $D = 0.4$?

Sol.: Since $D = 0.4 > min(0.7, 0.3) = 0.3$, we have that the corresponding rate is $R(0.4) = 0$ bits/ source symbol, so that the source can definitely be transmitted over the channel.

(c) Going back to the case $D = 0$, what is the largest rate r=source symbol/channel symbol that we can transmit over the channel?

Sol.: The condition reads

$$H(0.7) \cdot r \leq C \rightarrow r \leq \frac{C}{H(0.7)} = \frac{0.12}{0.88} = 0.14 \text{ source symb/ channel symb.}$$

(d) Repeat (c) with $D = 0.1$.

Sol.: We have

$$\begin{aligned} R(0.1) \cdot r &\leq C \rightarrow r \leq \frac{C}{R(0.1)} = \frac{0.12}{H(0.7) - H(0.1)} = \\ &= \frac{0.12}{0.88 - 0.47} = 0.29 \text{ source symb/ channel symb.} \end{aligned}$$

P2. (2 points) Consider the Gaussian channel in the figure, in which the transmitted signal X ($E[X^2] = P$) is received by two antennas with

$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= X + Z_2 \end{aligned}$$

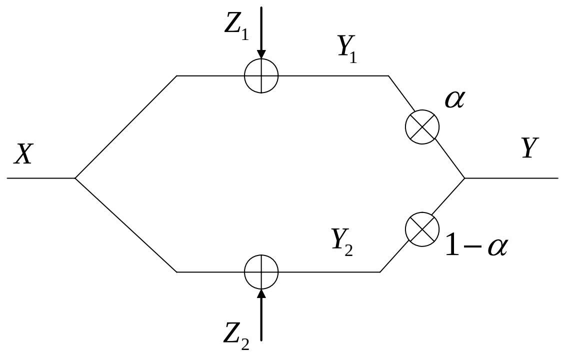

Figure 1:

where Z_1 and Z_2 are independent, and $E[Z_i^2] = N_i$ with $N_1 < N_2$. Moreover, the signal at the two antennas is combined as $Y = \alpha Y_1 + (1-\alpha)Y_2$ before detection ($0 \leq \alpha \leq 1$).

a) Find the capacity of the channel for a given α.

Sol.: Since we have

$$Y = \alpha Y_1 + (1-\alpha)Y_2 = X + \alpha Z_1 + (1-\alpha)Z_2,$$

the channel at hand is a Gaussian channel with signal power P and noise power $\alpha^2 N_1 + (1-\alpha)^2 N_2$ and the corresponding capacity is

$$C = \frac{1}{2}\log_2\left(1 + \frac{P}{\alpha^2 N_1 + (1-\alpha)^2 N_2}\right).$$

b) Using your result at the previous point, find the optimal α that maximizes the capacity and write down the corresponding maximum capacity.

Sol.: Maximizing the capacity C implies minimizing the noise power $\alpha^2 N_1 + (1-\alpha)^2 N_2$, which is a convex function of α, so that the optimality condition is

$$\frac{d}{d\alpha}(\alpha^2 N_1 + (1-\alpha)^2 N_2) = 0.$$

It follows that the optimal α reads

$$\alpha^* = \frac{N_2}{N_1 + N_2},$$

that is, proportionally more weight is given to the less noisy received signal. The corresponding capacity is

$$\begin{aligned} C &= \frac{1}{2}\log_2\left(1 + \frac{P}{\alpha^{*2} N_1 + (1-\alpha^*)^2 N_2}\right) = \\ &= \frac{1}{2}\log_2\left(1 + \frac{P}{N_1} + \frac{P}{N_2}\right). \end{aligned}$$

P3. (2 points) You are given a discrete memoryless multiple access channel (MAC) with inputs $X_1 \in \{0,1,2\}$ and $X_2 \in \{0,1,2\}$ and output $Y = X_1 + X_2$.

a) Assuming that both X_1 and X_2 are chosen uniformly and independently in their range, find the achievable rate region of the MAC.

Sol.: For fixed pmfs of the inputs X_1 and X_2, $p(x_1) = u(x_1)$ and $p(x_2) = u(x_2)$, we have

$$\begin{aligned} R_1 &\leq I(X_1;Y|X_2) = H(Y|X_2) - H(Y|X_1,X_2) = H(Y|X_2) = H(X_1) = \log_2 3 = 1.58 \\ R_2 &\leq I(X_2;Y|X_2) = H(X_2) = \log_2 3 = 1.58 \\ R_1 + R_2 &\leq I(X_1,X_2;Y) = H(Y) - H(Y|X_1,X_2) = H(Y) \end{aligned}$$

In order to evaluate $H(Y)$ we need to evaluate the pmf of Y as follows:

$$p(y) = \begin{cases} p_{X_1}(0)p_{X_2}(0) = 1/9 & y = 0 \\ 2 \cdot p_{X_1}(1)p_{X_2}(0) = 1/9 & y = 1 \\ 3 \cdot 1/9 & y = 2 \\ 2 \cdot 1/9 & y = 3 \\ 1/9 & y = 4 \end{cases}$$

so that

$$H(Y) = -2 \cdot \frac{1}{9} \log_2 \left(\frac{1}{9}\right) - 2 \cdot \frac{2}{9} \log_2 \left(\frac{2}{9}\right) - \frac{1}{3} \log_2 \left(\frac{1}{3}\right) = 2.2$$

b) Assume now that the two transmitters can cooperate for transmission over this MAC. What is the capacity region (notice that in this case maximization over the input distribution is required)?

Sol.: In this case we can achieve

$$R_1 + R_2 \leq I(X_1, X_2; Y) = H(Y)$$

for any joint pmf $p(x_1, x_2)$. In particular, we can optimize $p(x_1, x_2)$ in order to maximize $H(Y)$. We notice that $H(Y) \leq \log_2 |\mathcal{Y}| = \log_2 5 = 2.32$ with equality if $p(y) = u(y)$. Can this value be achieved with the right choice of $p(x_1, x_2)$? Observing that we must have

$$p(y) = \begin{cases} p(0,0) = 1/5 & y = 0 \\ p(0,1) + p(1,0) = 1/5 & y = 1 \\ p(1,1) + p(2,0) + p(0,2) = 1/5 & y = 2 \\ p(2,1) + p(1,2) = 1/5 & y = 3 \\ p(2,2) = 1/5 & y = 4 \end{cases},$$

we can conclude that it is enough to choose $p(x_1, x_2)$ as

$X_1 \backslash X_2$	0	1	2
0	1/5	1/10	1/15
1	1/10	1/15	1/10
2	1/15	1/10	1/5

We can then conclude that the capacity region with cooperation is characterized by $R_1 + R_2 \leq \log_2 5 = 2.32$.