ECE 755 - Digital communications
Final

Please provide clear and complete answers.

PART I: Questions -

Q.1. (1 point) Evaluate the approximate probability of error of the convolutional code characterized by the generator $G(D) = [1 \oplus D \;\, 1 \oplus D^2]$ as a function of E/N_0 (E is the energy per 2-dim) assuming 4-QAM transmission. What is the gain with respect to the corresponding uncoded system (with the same spectral efficiency)?

Sol.: The transition diagram is given by the following table (first column: starting state, second column: ending state, third column: input and output bits)

00	00	$(0, 00)$
00	10	$(1, 11)$
10	01	$(0, 10)$
10	11	$(1, 01)$
01	00	$(0, 01)$
01	10	$(1, 10)$
11	11	$(1, 00)$
11	01	$(0, 11)$

it easily follows that the minimum-Hamming weight error event is (00, 10, 01, 00), whose Hamming weight is 4. Therefore, the Euclidean distance is $4 \cdot 4c^2 = 16c^2$ (assuming Gray mapping) and the approximate probability of error (for both sequence and symbol) is ($E = 2c^2$)

$$P_e \simeq KQ\left(\sqrt{\frac{16c^2}{2N_0}}\right) = KQ\left(\sqrt{\frac{4E}{N_0}}\right).$$

The corresponding uncoded system is BPSK, whose probability of error is

$$P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right),$$

so that the gain is $3dB$.

Q.2. (1 point) Compare the performance at the point above with the approximate probability of error for the recursive convolutional code $G(D) = [1 \;\, \frac{1\oplus D^2}{1\oplus D}]$ (you are free to choose the tail bits) when the bit sequence of all ones is transmitted.

Sol.: The two codes are equivalent. Moreover, since the code is linear, any two codewords have the same conditional probability of error. It follows that the requested approximate probability of error is the same as at the previous point.

Q3. (1 point) Consider an analog PLL with loop filter

$$L(s) = K_0 \frac{s+1}{s+K}.$$

a. We are interested in obtaining a lock-in range of $0.5kHz$ and a bandwidth of $50Hz$. Find reasonable values for K_0 and K to satisfy these constraints.
b. Assume that the input is $\theta(t) = 2\pi f_0 t + \theta$. Find the asymptotic phase error.

Sol.: **a.** For the lock-in range:

$$\begin{aligned} |f_0| &\leq \frac{|L(0)|}{2} = 0.5kHz \\ &\Longrightarrow \frac{K_0}{K} = 1000 \end{aligned}$$

while the bandwidth is well approximated by $\sqrt{K_0}/2\pi$ so that

$$K_0 = (2\pi \cdot 50)^2 = \pi^2 \cdot 10^4$$

It follows that

$$K = \pi^2 \cdot 10.$$

It can be checked from the transfer function

$$\frac{\Phi(s)}{\Theta(s)} = \frac{L(s)}{L(s)+s} = \frac{K_0(s+1)}{s^2 + (\pi^2 \cdot 10 + \pi^2 \cdot 10^4)s + \pi^2 \cdot 10^4}$$

that the PLL is stable.
b. Using the final value theorem, we have (recall that the Laplace transform of the input is $\Theta(s) = 2\pi f_0/s^2 + \theta/s$

$$\begin{aligned} \lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} s\left(\frac{2\pi f_0}{s^2} + \frac{\theta}{s}\right) \frac{s(s+K)}{s^2 + (K+K_0)s + K_0}, \end{aligned}$$

where we have used the fact that the transfer function for the error is

$$\frac{1}{1 + L(s)/s} = \frac{s(s+K)}{s^2 + (K+K_0)s + K_0}.$$

Following the calculations above, we get

$$\lim_{t\to\infty} e(t) = \frac{2\pi f_0 K}{K_0} = \frac{2\pi f_0}{1000}.$$

Q4. (1 point) Consider the linear code characterized by the generator matrix **G** =[1 1 1] with BPSK modulation. Assume that the received signal is [1.2 − 0.01 2.1], the energy per 2 dim is $E = 1$ and the noise power per dimension is $\sigma^2 = 3$. Run the message passing algorithm until convergence (do you expect it to converge?) and find the corresponding log-likelihoods at the bit nodes.

Sol.: See class notes for the steps of the algorithm. The log-likelihoods at the check nodes after convergence coincide with the real values (due to the fact that the Tanner graph has no cycles) and are equal to

$$\begin{aligned}\lambda_i &= \frac{2\sqrt{E}}{\sigma^2}(r_1 + r_2 + r_3) \\ &= \frac{2}{3}(1.2 - 0.01 + 2.1) \\ &= 2.19.\end{aligned}$$

PART II: Problems -

P.1. (2 points) Two symbols a_0, $a_1 \in \mathcal{A} = \{-1, 1\}$ are transmitted over an ISI channel that is described by the transfer function $M(z) = 1 - 0.2z^{-1}$. The a priori probabilities of the two bits are $p_{A_0}(-1) = 1/2$ and $p_{A_1}(-1) = 1/4$. We would like to obtain reliability information about the transmitted symbols in the form of log-likelihood ratios given that the received signal is $\mathbf{z}$ =[0.2 0.1 −0.3] and knowing that the noise power is $\sigma^2 = N_o/2 = 0.2$ (and that we have $a_k = -1$ for $k < 0$ and $k \geq 2$).
a. Draw the trellis describing the possible transmitted sequences with appropriate branch metrics $\gamma_k(p, q)$ for the BCJR algorithm.
b. Perform the BCJR algorithm and find the a posteriori probabilities of the transmitted symbols ($p_{A_k}(a|\mathbf{z})$). What are the log-likelihood ratios for the three symbols? Which symbol can be decided on with the highest confidence?

Sol.: **a.** The branch metrics $\gamma_k(p, q)$ can be written as

$$\begin{aligned}\gamma_k(p, q) &= p_{A_k}(a) f(z_k | \Psi_k = p, \Psi_{k+1} = p) \\ &= p_{A_k}(a) \exp(-(z_k - s^{(p,q)})/(2\sigma^2)) \\ &= p_{A_k}(a) \exp(-(z_k - s^{(p,q)})/(0.4))\end{aligned}$$

where we have used standard notation and have dropped the constant $1/\sqrt{2\pi\sigma^2}$ since it is common to all terms and thus immaterial for the algorithm. The pair of transmitted symbol a_k and received signals $s^{(p,q)}$ corresponding to state transition (p, q) can be easily found, and so can the branch metrics $\gamma_k(p, q)$, see solutions from previous year or class notes.

P.2. (2 points) The equivalent discrete-time ISI channel experienced by a given communication system is given by $H(z) = 1 + 4z^{-1}$ with noise power spectral density $S_n(z) = N_0 = 0.8$.

a. Design the ZF-LE. Can the equalizer be realized? Find the numerical value of the corresponding probability of error for a BPSK constellation assuming energy per bit $E_b = 1/4$.

b. Repeat the point above for the ZF-DFE.

c. Consider implementing the ZF-DFE filter at the point above via transmitter precoding. Draw the block diagram of the corresponding system assuming 4-PAM transmission (with alphabet $\{-3,-1,1,3\}$). Moreover, find the transmitted sequence x_k ($k = 0,1,2,3$) given the input symbols $\mathbf{a} = [3,-3,3,1]$.

a. The channel $H(z) = 1 + 4z^{-1}$ can be written as

$$H(z) = 4z^{-1}(1 + 0.25z) = z^{-1}H_o \cdot H_{\max}(z).$$

The term z^{-1} only implies a delay of one unit time in the decision and is thus not further accounted for below: we set without loss of generality $H(z) = H_o \cdot H_{\max}(z)$ with $H_o = 4$ and $H_{\max}(z) = (1 + 0.25z)$. The ZF-LE is defined by the filter

$$C(z) = \frac{1}{H_o \cdot H_{\max}(z)} = \frac{1}{4(1 + 0.25z)},$$

which is clearly anti-causal stable (not realizable) having a pole in $z = -4$.
The probability of error is obtained by deriving the noise power at the output of the equalizer:

$$\varepsilon^2_{ZF-LE} =< \frac{N_0}{16|1 + 0.25e^{i\omega}|^2} >_{A,(-\pi,\pi)} = \frac{N_0}{16}\sum_{k=0}^{\infty}(0.25)^{2k} = \frac{N_0}{16}\frac{1}{1-\frac{1}{16}} = \frac{N_0}{15} = \frac{0.8}{15},$$

and then using the expression for the probability of error with BPSK:

$$P_{e,ZF-LE} = Q\left(\sqrt{\frac{2E_b}{\varepsilon^2_{ZF-LE}/2}}\right) = Q\left(\sqrt{\frac{1/2}{0.4}15}\right) = 7 \cdot 10^{-6}.$$

b. The pre-cursor equalizer is given by the all-pass filter

$$\frac{1}{H_0}\frac{H^*_{\max}(1/z^*)}{H_{\max}(z)} = \frac{1}{4}\frac{1 + 0.25z^{-1}}{1 + 0.25z},$$

which is clearly anti-causal stable (not realizable). The feedback filter is instead given by

$$H^*_{\max}(1/z^*) - 1 = 0.25z^{-1}.$$

The noise power at the input of the decision device is:

$$\varepsilon^2_{ZF-DFE} =< \frac{N_0}{16|1 + 0.25e^{i\omega}|^2} >_{G,(-\pi,\pi)} = \frac{N_0}{16} = 0.05,$$

so that

$$P_{e,ZF-DFE} = Q\left(\sqrt{\frac{1/2}{0.05/2}}\right) = Q\left(\sqrt{20}\right) = 3 \cdot 10^{-6}.$$

c. The block diagram can be found in the textbook: it consists of the feedback loop moved to the transmitter side with the addition of a mod-8 operation in lieu of the decision device in the loop. The transmitted sequence is given by:

$$\begin{aligned}
x_0 &= \text{mod}_8(a_0) = 3\\
x_1 &= \text{mod}_8(a_1 - 0.25x_0) = \text{mod}_8(-3.75) = -3.75\\
x_2 &= \text{mod}_8(a_2 - 0.25x_1) = \text{mod}_8(3.94) = 3.94\\
x_3 &= \text{mod}_8(a_3 - 0.25x_2) = \text{mod}_8(0.015) = 0.015.
\end{aligned}$$

P.3. You need to design a system that transmits $4Mbits/s$ over a bandwidth of $1.2MHz$ and with a roll-off factor of $\alpha = 0.2$ by using TCM codes. The goal is obtaining an overall gain of about $3dB$ (in terms of E/N_0) with respect to the corresponding uncoded system with the same spectral efficiency.

a. Find the necessary spectral efficiency and constellation size (you can use "cross-constellations" in case an M-QAM constellation does not exist with the necessary number of points). Also, specify the number of states and the constraint length you would choose for the convolutional code.

b. Sketch the block diagram of the proposed TCM code. In particular, how many bits should be coded? To answer this question, derive the necessary set partitioning and the corresponding minimum distance that guarantee the requested gain of $3dB$.

c. Repeat the problem for a gain of $5dB$.

Sol.: **a.** Spectral efficiency:

$$\nu = \frac{4Mbits/s}{1.2MHz} = \frac{4}{1.2}$$

and number of bits per 2-dim:

$$\rho = \nu(1+\alpha) = 4.$$

Therefore, the constellation size is

$$2 \cdot 2^{\rho} = 32,$$

so that we can pick a 32-cross constellation (see textbook and notes).

Following Ungerboeck's rule of thumb, a gain of about $3dB$ should be guarantee with a convolutional encoder with four states and thus constraint length equal to 3.

b. We need to check the level of partitioning of the constellation that is necessary to guarantee a gain of $3dB$. Defining as $[0, c]$ and $[c, 0]$ the vectors generating the lattice on which the constellation is built, it can be easily checked (see also notes) that the minimum distance corresponding to the first level of partitioning is $2\sqrt{2}c$, for the second $4c$, for the third $4\sqrt{2}c$and for the fourth $4\sqrt{2}c$. Relating c to the energy per two dimensions E, we obtain (by calculating the average energy of the constellation assuming equal probability points):

$$E = 20c^2.$$

Now, the reference system is uncoded 16QAM for which the minimum distance is

$$d^2_{\min,16QAM} = 4c^2 = 4E/10 = 2E/5,$$

where the latter equality follows from the fact that the average energy for 16QAM is $E = 10c^2$. In order to obtain a gain of 3dB, the minim distance should be $d^2_{\min} \geq 4E/5$. Without coding, the minimum distance squared of the 32-cross constellation would be $8/20E = 2/5E$, which is not enough (not surprisingly). But with TCM and two levels, we obtain $16/20 = 4/5E$, which is the desired result. As such, the number of coded bits need to be one.

c. Since we want a gain of 5dB, from Ungerbroeck's rule, we would need 16 states and thus a constraint length of 5. Moreover, we need to encode 2 bits, in order to obtain a minimum distance squared of $32/20E = 8/5E$, which provides a gain of $4 = 6dB$.