ECE 673-Random signal analysis I
Final - Dec. 20th 2006

Q1. The random processes $X[n]$ and $Y[n]$ are both WSS with zero mean and autocorrelation sequences $r_X[k]$ and $r_Y[k]$. Moreover, every sample of $X[n]$ is independent from every sample of $Y[n]$. Is $Z[n] = X[n] + Y[n]$ WSS? If so, find mean and autocorrelation sequence of $Z[n]$.

Sol.: Let us calculate mean and autocorrelation in order to check if the process is WSS:

$$E[Z[n]] = E[X[n]] + E[Y[n]] = 0$$

$$\begin{aligned} E[Z[n]Z[n+k]] &= E[(X[n]+Y[n])(X[n+k]+Y[n+k])] = \\ &= E[X[n]X[n+k]] + E[Y[n]Y[n+k]] = \\ &= r_X[k] + r_Y[k] = r_Z[k], \end{aligned}$$

where we have used the fact that $E[X[n]Y[n+k]] = E[X[n]] \cdot E[Y[n+k]] = 0$ since $X[n]$ and $Y[n]$ are independent and zero mean. We can conclude that the process is in fact WSS.

Q2. Consider the same random process $X[n]$ and $Y[n]$ of the previous point. Is $Z[n] = X[n]Y[n]$ WSS? If so, find mean and autocorrelation sequence of $Z[n]$.

Sol.: Let us calculate mean and autocorrelation in order to check if the process is WSS. We have

$$E[Z[n]] = E[X[n]] \cdot E[Y[n]] = 0,$$

since the processes are independent and zero mean, and

$$\begin{aligned} E[Z[n]Z[n+k]] &= E[(X[n]Y[n]) \cdot (X[n+k]Y[n+k])] = \\ &= E[X[n]X[n+k]] \cdot E[Y[n]Y[n+k]] = \\ &= r_X[k] \cdot r_Y[k] = r_Z[k], \end{aligned}$$

where we have used the fact that $X[n]X[n+k]$ and $Y[n]Y[n+k]$ are independent. We can conclude that the process is in fact WSS.

Q3. You are given a WSS process $U[n]$ with power spectrum density $P_U(f) = 1/(2 - 2\cos(4\pi f))$ and you are asked to design a LTI with impulse response $h[n]$ such that the output of the LTI with input $U[n]$ is white noise. Find the impulse response $h[n]$.

Sol.: The output of the LTI with impulse response $h[n]$ and input $U[n]$ is a WSS process with power spectral density

$$P_X(f) = |H(f)|^2 P_U(f) = \frac{|H(f)|^2}{(2 - 2\cos(4\pi f))}.$$

Therefore, in order for the output $X[n]$ to be white noise, we should set $|H(f)|^2 = (2 - 2\cos(4\pi f))$, possibly multiplied by an arbitrary constant. This implies that $H(f) = 1 -$

$\exp(-j4\pi f)$ (in fact: $|H(f)|^2 = H(f)H^*(f) = (1-\exp(-j4\pi f))(1-\exp(j4\pi f)) = 1+1-2\cos(4\pi f))$ and thus $h[n] = \delta[n] - \delta[n-2]$.

Q4. A WSS random process $X[n]$ is characterized by mean $\mu_X = 1$ and autocorrelation function $r_X[k] = (1/2)^{|k|}$. Find the covariance $cov(X[1], X[2])$.

Sol.: We have

$$\begin{aligned} cov(X[1], X[2]) &= E[X[1]X[2]] - E[X[1]]E[X[2]] = \\ &= r_X[1] - \mu_X^2 = 1/2 - 1 = -1/2. \end{aligned}$$

P1. Consider a Linear Time-Invariant system (LTI) with input $U[n]$ being a white Gaussian noise with power $\sigma^2 = 1/2$ and impulse response $h[n] = \delta[n] + \delta[n-1]$.

(i) Is the output process $X[n]$ WSS? Is it a Gaussian process? Is it stationary? If it is WSS, find mean and autocorrelation of the output.

Sol.: Yes, $X[n]$ is WSS and Gaussian, since it is the output of a LTI system with a WSS Gaussian process at the input. Let us evaluate mean and autocorrelation

$$\mu_X = \mu_U \sum_n h[n] = 0,$$

$$r_X[k] = \frac{1}{2} r_h[k] = \frac{1}{2}(2\delta[n] + \delta[n-1] + \delta[n+1]),$$

where the latter result can be easily shown by using the graphical approach.

(ii) Find *and sketch* the joint PDF of $X[3]$ and $X[4]$.

Sol.: From the previous point, $X[3]$ and $X[4]$ are jointly Gaussian with mean and covariance matrix as ($\mathbf{X} = [X[3]\ X[4]]^T$)

$$\mathbf{X} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} r_X[0] & r_X[1] \\ r_X[-1] & r_X[0] \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \right).$$

Therefore, $X[3]$ and $X[4]$ are *standard* jointly Gaussian variables with $\rho = 0.5$.

(iii) Find the conditional PDF of $X[4]$ given $X[3]$.

Sol.: Since $X[3]$ and $X[4]$ are standard jointly Gaussian variables, we have that

$$(X[4] \mid X[3] = x) \sim \mathcal{N}(\rho x = 0.5x, 1 - \rho^2 = 0.75).$$

(iv) Evaluate the probability that $X[3] - X[4] > 1$.

Sol.: Linear combinations of jointly Gaussian random variables are Gaussian as well. In particular, it is zero-mean and with variance

$$\begin{aligned} var(X[3] - X[4]) &= var(X[3]) + var(-X[4]) + 2 \cdot cov(X[3], -X[4]) = \\ &= var(X[3]) + var(X[4]) - 2 \cdot cov(X[3], X[4]) = \\ &= 2 \cdot 1 - 2 \cdot 0.5 = 1 \end{aligned}$$

so that $X[3] - X[4] \sim \mathcal{N}(0,1)$ and

$$P[X[3] - X[4] > 1] = Q(1).$$

P2. A process is described by the difference equation

$$X[n] = \frac{1}{2}X[n-2] + U[n],$$

where $U[n]$ is white noise with variance one.

(i) Interpret the difference equation as a LTI by finding the corresponding impulse response $h[n]$.

Sol.: Let us find the impulse response corresponding to the difference equation at hand. We can do so by letting $U[n] = \delta[n]$ and evaluating the output $h[n]$ time instant by time instant (notice that $h[n] = 0$ for $n < 0$)

$$\begin{aligned}
h[0] &= \delta[0] = 1 \\
h[1] &= \delta[1] = 0 \\
h[2] &= \frac{1}{2}h[0] = \frac{1}{2} \\
h[3] &= \frac{1}{2}h[1] = 0 \\
h[4] &= \frac{1}{2}h[2] = \frac{1}{4} \\
&\cdots
\end{aligned}$$

We can then conclude that

$$h[n] = \begin{cases} 0 & n \text{ odd} \\ (\frac{1}{2})^{n/2} & n \text{ even} \end{cases}.$$

(ii) Evaluate and plot the power spectral density of $X[n]$.

Sol.: In order to find the power spectra density it is convenient to use the z-transform (system function) approach starting from the difference equation:

$$\begin{aligned}
X[n] &= \frac{1}{2}X[n-2] + U[n] \\
&\longleftrightarrow X(z) = \frac{1}{2}X(z)z^{-2} + U(z),
\end{aligned}$$

which easily implies

$$H(z) = \frac{1}{1 - 1/2 \cdot z^{-2}}.$$

Therefore, the frequency response reads

$$H(f) = \frac{1}{1 - 1/2 \cdot \exp(-4\pi f)}.$$

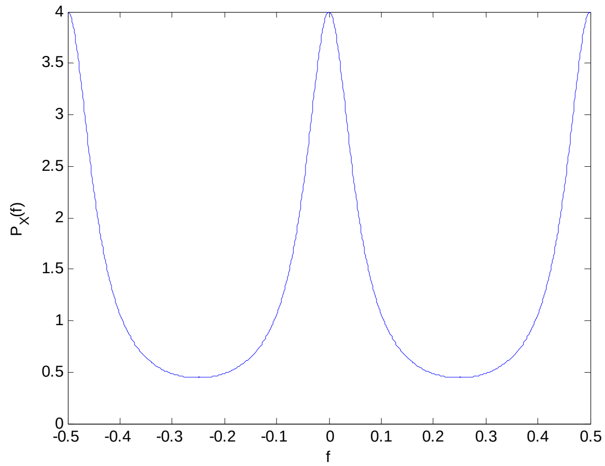

Figure 1:

The power spectral density of the output is then

$$\begin{aligned} P_X(f) &= |H(f)|^2 = \frac{1}{(1-1/2\cdot\exp(-4\pi f))(1-1/2\cdot\exp(4\pi f))} = \\ &= \frac{1}{1+1/4-\cos(4\pi f)} = \frac{1}{5/4-\cos(4\pi f)}. \end{aligned}$$

P3. We are interested in studying the sum-process

$$X[n] = \sum_{i=0}^{n} U[i],$$

where $U[n]$ is an IID process with $U[n] \sim Ber(0.6)$.

(i) Is $X[n]$ stationary? Is it WSS?

Sol.: No, the process is not stationary and not even WSS. In fact, the average is dependent on time n:

$$E[X[n]] = \sum_{i=0}^{n} E[U[n]] = (n+1)E[U[n]] = (n+1)\cdot 0.6.$$

(ii) Evaluate the probabilities $P[X[0]=1]$ and $P[X[0]=1, X[1]=2]$.

Sol.: We have

$$P[X[0]=1] = P[U[0]=1] = 0.6$$

and

$$\begin{aligned} P[X[0] &= 1, X[1]=2] = P[U[0]=1, U[1]=1] = \\ P[U[0] &= 1]\cdot P[U[1]=1] = (0.6)^2 = 0.36. \end{aligned}$$

(iii) Find the PMF of $X[n]$ for a given n.

Sol.: Since $X[n]$ is a sum of $n+1$ independent Bernoulli random variables, $X[n] \sim bin(n+1, 0.6)$.