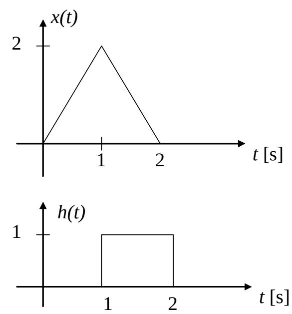
ECE 232 - Circuits and Systems II
Final (Spring 2011)

Please provide clear and complete answers. Don't forget to specify the units of measure!

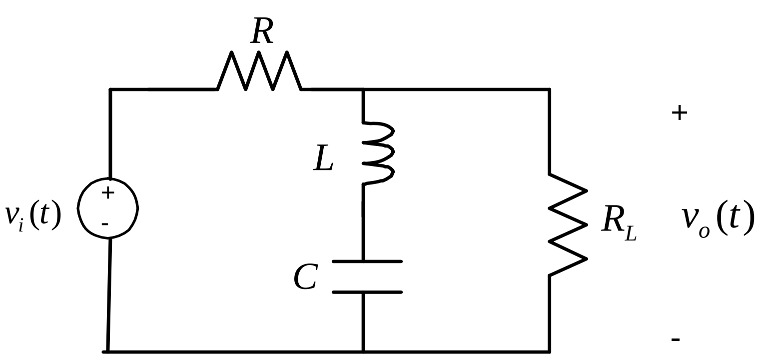
1. (2 points) Consider the transfer function

$$H(s) = 100\frac{s(s+1)}{(s+10)(s+100)(s+1000)}.$$

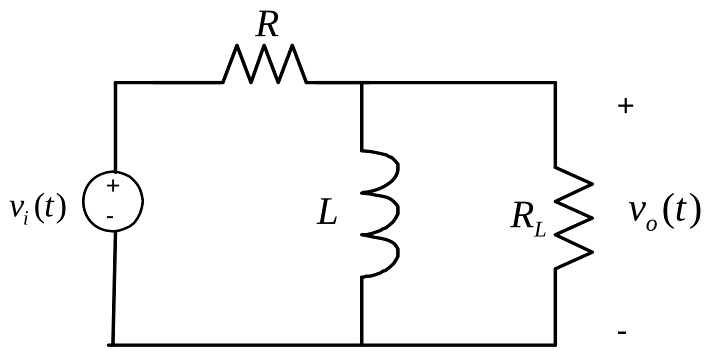
a. Draw the amplitude Bode plot for the transfer function. Specify clearly the relevant values on the two axes. In particular, what is (approximately) $H_{\max}$?
b. What type of filter is it? Using Bode's approximation, find the $3dB$ cut-off frequency (or frequencies) (Hint: a linear increase of $20dB$ for every decade is the same as an increase of $3dB$ for every multiplicative increase by $\sqrt{2}$).



2. (2 points) Perform the convolution $y(t) = x(t) * h(t)$ of the two signals $x(t)$ and $h(t)$ shown in the figure on the left.



3. (3 points) Consider the circuit in the figure on the left with $R = 1k\Omega$, $L = 100mH$, $C = 100pF$ and $R_L = 10k\Omega$.
a. Find the transfer function between $v_i(t)$ and $v_o(t)$.
b. Find and sketch the amplitude response. What type of filter is it? What is $H_{\max}$ and what are the $3dB$ cut-off frequencies?
c. Discuss the effect of R_L.
d. Calculate the quality factor and relate this result to the position of the poles in the complex plane.



4. (3 points) Consider the circuit in the figure on the left with $R = 1M\Omega$, $L = 1mH$, $R_L = 1M\Omega$.
a. Find the transfer function between $v_i(t)$ and $v_o(t)$. What are the poles and zeros?
b. Sketch the amplitude and phase response. What type of filter is it? What is $H_{\max}$ and what is the $3dB$ cut-off frequency?
c. Assume that the inductor has a current at time zero equal to $1mA$ (with direction from top to bottom) and that $v_i(t) = e^{-1000t}u(t)$. Find the Laplace transform of the output $v_o(t)$.

Sol.:
1. We fist write

$$\begin{aligned} H(s) &= 100\frac{s(1+s)}{10\cdot 100\cdot 1000(1+\frac{s}{10})(1+\frac{s}{100})(1+\frac{s}{1000})} \\ &= \frac{1}{10^4}\frac{s(1+s)}{(1+\frac{s}{10})(1+\frac{s}{100})(1+\frac{s}{1000})} \end{aligned}$$

a. Using the approach seen in class, the Bode plots starts with a $20dB/decade$ slope until frequency $1rad/s$, then the slope is $40dB/decade$ until $10rad/s$, where it becomes $20dB/decade$, to become $0dB/decade$ at frequency $100rad/s$ and finally $-20dB/decade$ at $1000rad/s$. Note that H_{max} is approximately obtained between $100rad/s$ and $1000rad/s$, so we get

$$H_{\max} \simeq |H(j100)| \simeq \frac{1}{10^4}\frac{100\cdot 100}{10} = 0.1.$$

b. The filter is bandpass with central frequency between $100rad/s$ and $1000rad/s$. The $3dB$ cut-off frequencies are approximately

$$\begin{aligned} \omega_{c1} &= \frac{100}{\sqrt{2}} = 70.71rad/s \\ \omega_{c2} &= 1000\sqrt{2} = 1414.2rad/s. \end{aligned}$$

2. We perform the convolution between $x(t)$ and $h(t+1)$ (that is, $h(t)$ shifted left by 1 second) as $\tilde{y}(t) = x(t) * h(t+1)$. The final result $y(t)$ will be obtained by moving the obtained solution $\tilde{y}(t)$ to the right by 1 second.

As for the convolution $\tilde{y}(t) = x(t) * h(t+1)$ we get

$$\begin{aligned}
\tilde{y}(t) &= \frac{2t^2}{2} = t^2 \ \text{ for } 0 \leq t \leq 1 \\
\tilde{y}(t) &= \left(1 - \frac{2(t-1)^2}{2}\right) + \left(1 - \frac{2(2-t)^2}{2}\right) \\
&= 2 - (t-1)^2 - (2-t)^2 \\
&= 6t - 2t^2 - 3 \ \text{ for } 1 \leq t \leq 2 \\
\tilde{y}(t) &= (2 - (t-1))^2 \\
&= t^2 - 6t + 9 \text{ for } 2 \leq t \leq 3 \\
\tilde{y}(t) &= 0 \text{ for } t \leq 0 \text{ and } t \geq 3.
\end{aligned}$$

In the calculations above, we simply used the formula for the area of a triangle. We conclude that the desired convolution is

$$\begin{aligned}
y(t) &= (t-1)^2 \ \text{ for } 1 \leq t \leq 2 \\
y(t) &= 2 - (t-2)^2 - (3-t)^2 \ \text{ for } 2 \leq t \leq 3 \\
y(t) &= (2 - (t-2))^2 \ \text{ for } 3 \leq t \leq 4 \\
y(t) &= 0 \text{ for } t \leq 1 \text{ and } t \geq 4.
\end{aligned}$$

3. a.The transfer function is, by the voltage division rule, given by

$$H(s) = \frac{Z_{eq}(s)}{R + Z_{eq}(s)},$$

where

$$\begin{aligned}
Z_{eq}(s) &= R_L || \left(sL + \frac{1}{sC}\right) \\
&= \frac{R_L\left(sL + \frac{1}{sC}\right)}{R_L + sL + \frac{1}{sC}},
\end{aligned}$$

so that

$$\begin{aligned}
H(s) &= \frac{R_L\left(sL + \frac{1}{sC}\right)}{R(R_L + sL + \frac{1}{sC}) + R_L\left(sL + \frac{1}{sC}\right)} \\
&= \frac{R_L\left(s^2LC + 1\right)}{R(R_L sC + s^2LC + 1) + R_L\left(s^2LC + 1\right)} \\
&= \frac{\frac{R_L}{R+R_L}\left(s^2 + \frac{1}{LC}\right)}{s^2 + \frac{R_L}{R_L+R}\frac{R}{L}s + \frac{1}{LC}} \\
&= \frac{K(s^2 + \omega_o^2)}{s^2 + 2K\alpha s + \omega_o^2},
\end{aligned}$$

where we have defined $\omega_o = \frac{1}{\sqrt{LC}} = \sqrt{10^{11}} = 316,228$ rad/s, $K = \frac{R_L}{R_L+R} = \frac{10}{11}$, $\alpha = \frac{R}{2L} = 0.5 \cdot 10^4$ rad/s. Note that ω_o and α are the usual parameters for a series RLC circuit.

Now, define for convenience $\alpha' = K\alpha$. We then have

$$H(s) = K\frac{(s^2 + \omega_o^2)}{s^2 + 2\alpha' s + \omega_o^2}.$$

b. From the equation above, we see that the filter is bandreject. The amplitude response is

$$H(s) = K\frac{(-\omega^2 + \omega_o^2)}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\alpha'^2\omega^2}}.$$

From this, we see that $H_{\max} = K = \frac{10}{11} = 0.9091$, and the cut-off frequencies are

$$\begin{aligned}
\omega_{c1} &= \alpha + \sqrt{\alpha^2 + \omega_0^2} = -0.5 \cdot 10^4 + \sqrt{(0.5 \cdot 10^4)^2 + 10^{11}} \\
&= 3.212 \times 10^5 \text{ rad/s} \\
\omega_{c2} &= -\alpha + +\sqrt{\alpha^2 + \omega_0^2} = 3.112\,7 \times 10^5 \text{ rad/s}.
\end{aligned}$$

c. The load resistance R_L reduces $H_{\max}$ (from 1 to 0.9091) and reduces the bandwidth (from 10^4 rad/s to $0.9091 \cdot 10^4$ rad/s) with respect to the filter with infinite load resistance.
d. The quality factor is

$$Q = \frac{\omega_o}{\beta} = \frac{316228}{0.9091 \cdot 10^4} = 34.785 > 0.5,$$

which implies that the circuit works in the underdamped regime and the poles are complex conjugate.

4. a. The transfer function is

$$\begin{aligned}
H(s) &= \frac{R_L||sL}{R_L||sL + R} \\
&= \frac{R_L sL}{R_L sL + R(R_L + sL)} \\
&= \frac{s\frac{R_L}{R_L+R}}{s + \frac{R}{L}\frac{R_L}{R_L+R}} \\
&= K\frac{s}{s + \frac{K}{\tau}}
\end{aligned}$$

with $K = \frac{R_L}{R_L+R} = 0.5$ and $\tau = \frac{L}{R} = 10^{-9}$ sec. We have a zero in $s = 0$ and a pole in $s = -0.5 \cdot 10^{-9}$.
b. Please see book and notes for the plots. We have $H_{\max} = 0.5$ and $\omega_c = \frac{K}{\tau} = 0.5 \cdot 10^9$ rad/s. It is a high-pass filter.
c. Using the superposition principle, we have that

$$V_o(s) = H(s)V_i(s) + H_1(s)V_1(s),$$

where

$$V_i(s) = \frac{1}{s + 1000},$$

and $V_1(s)$ is the equivalent voltage source to be added in series to the inductor to account for the initial conditions and $H_1(s)$ is the corresponding transfer function to the output. We have

$$V_1(s) = LI_o = 10^{-6}$$

and

$$\begin{aligned} H_1(s) &= -\frac{R||R_L}{R||R_L + sL} \\ &= -\frac{RR_L/(L(R+R_L))}{RR_L/L(R+R_L)+s} \\ &= -\frac{\frac{K}{\tau}}{s+\frac{K}{\tau}}. \end{aligned}$$