ECE 755 - Digital communications
Final

Please provide clear and complete answers.

PART I: Questions -

Q.1. Is this the set $\mathcal{C} = \{100, 010, 110\}$ a linear block code? Why? If not, complete the set so that it is. Write generator matrix, parity check matrix and find the minimum Hamming distance of the resulting code.

Sol.: No, because it does not contain the all-zero codeword. The matrices are:

$$\begin{aligned} \mathbf{G} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ \mathbf{H} &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

and the minimum Hamming distance is 1 (not a very good code!).

Q.2. A codeword is generated by concatenating a $(2,1)$ repetition code, followed by an interleaver and a simple recursive convolutional code $1/(1 \oplus D)$ initialized to a zero state (see Figure below). Consider two information bits at the input (and thus four bits at the output of the repetition code). Moreover, assume that the interleaver changes the order of the bits as $\pi = [1\ 3\ 2\ 4]$ (that is, the first output bit is the first bit at the input, the second output bit is the third at the input and so on). Consider all the possible first four bits at the output of the convolutional code as a code. What is its rate? Find a parity check matrix describing the code and the corresponding Tanner's graph.

Sol.: The rate is clearly $R = 2/4 = 1/2$. Let us define the two input bits as $b_1\ b_2$. After the interleaver we thus have $b_1\ b_2\ b_1\ b_2$. The coded bits are then:

$$\begin{aligned} c_1 &= b_1 \\ c_2 &= b_2 \oplus c_1 \\ c_3 &= b_1 \oplus c_2 \\ c_4 &= b_2 \oplus c_3. \end{aligned}$$

To find a parity check matrix, we need to eliminate $b_1\ b_2$ from the equations above. For instance, from the first and third we get $c_1 \oplus c_2 \oplus c_3 = 0$, while from the second and third $c_1 \oplus c_2 \oplus c_3 \oplus c_4 = 0$. A suitable parity check matrix is then

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We can check that this leads to the correct code also by enumerating all the 4 codewords composing the code: $\mathcal{C} = \{0000, 0110, 1100, 1010\}$ and verifying that $\mathbf{c}\mathbf{H}^T = 0$ for $\mathbf{c} \in \mathcal{C}$.

Q3. Consider a scaled lattice $\alpha \cdot \Lambda$, obtained by multiplying all the basis vectors of the lattice Λ by $\alpha > 0$. Show that the coding gain of the scaled lattice is equal to that of the original lattice: $\gamma_{\alpha \cdot \Lambda} = \gamma_{\Lambda}$.

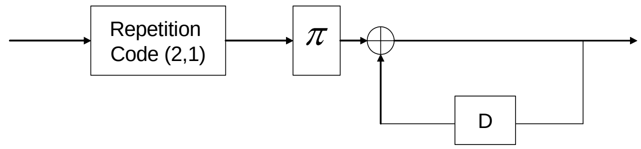

Figure 1:

Sol: By definition, we have:

$$\gamma_\Lambda = \frac{d^2_{\min}(\Lambda)}{V(\Lambda)^{2/n}}.$$

For the scaled lattice:

$$\gamma_{\alpha\cdot\Lambda} = \frac{d^2_{\min}(\alpha\Lambda)}{V(\alpha\Lambda)^{2/n}} = \frac{\alpha^2 d^2_{\min}(\Lambda)}{(\alpha^n V(\Lambda))^{2/n}} = \frac{d^2_{\min}(\Lambda)}{V(\Lambda)^{2/n}} = \gamma_\Lambda.$$

Q4. Consider a 4-PAM constellation $x \in \{-3, -1, 1, 3\}$ where mapping from the bit stream occurs as $\{10, 11, 01, 00\}$ (that is, "10" corresponds to "-3", "11" to "-1" etc.). Having received signal $y = x + n = -2.5$ with $n \sim N(0, 0.1)$, find the log-likelihood ratio of the a posteriori probability of the first of the two bits mapping the transmitted 4-PAM symbol (assume that the bits are a priori equally likely).

Sol.: By direct calculation:

$$\begin{aligned}
\log\frac{\Pr[b_1=1|y]}{\Pr[b_1=0|y]} &= \log\frac{\Pr[x=-3|y]+\Pr[x=-1|y]}{\Pr[x=1|y]+\Pr[x=3|y]} = \\
&= \log\frac{\Pr[y|x=-3]+\Pr[y|x=-1]}{\Pr[y|x=1]+\Pr[y|x=3]} = \\
&= \log\frac{e^{-(y+3)^2/2\sigma^2}+e^{-(y+1)^2/2\sigma^2}}{e^{-(y-1)^2/2\sigma^2}+e^{-(y-3)^2/2\sigma^2}} = \\
&= \log\frac{e^{-(-2.5+3)^2/4}+e^{-(-2.5+1)^2/4}}{e^{-(-2.5-1)^2/4}+e^{-(-2.5-3)^2/4}} = 60.
\end{aligned}$$

PART II: Problems -

P.1. The equivalent discrete-time ISI channel experienced by a given communication system is given by $H(z) = 1 + 2z^{-1}$ with noise power spectral density $S_n(z) = N_0 = 0.8$.

a. Design the ZF-LE. Can the equalizer be realized? Find the numerical value of the corresponding probability of error for a BPSK constellation assuming energy per bit $E_b = 1$.

b. Repeat the point above for the ZF-DFE.

c. Consider implementing the ZF-DFE filter at the point above via transmitter precoding. Draw the block diagram of the corresponding system assuming 4-PAM transmission (with alphabet $\{-3, -1, 1, 3\}$). Moreover, find the transmitted sequence x_k ($k = 0, 1, 2, 3$) given the input symbols $\mathbf{a} = [3, -3, 3, 1]$.

a. The channel $H(z) = 1 + 2z^{-1}$ can be written as

$$H(z) = 2z^{-1}(1 + 0.5z) = z^{-1} H_o \cdot H_{\max}(z).$$

The term z^{-1} only implies a delay of one unit time in the decision and is thus not further accounted for below: we set without loss of generality $H(z) = H_o \cdot H_{\max}(z)$ with $H_o = 2$ and $H_{\max}(z) = (1 + 0.5z)$. The ZF-LE is defined by the filter

$$C(z) = \frac{1}{H_o \cdot H_{\max}(z)} = \frac{1}{2(1 + 0.5z)},$$

which is clearly anti-causal stable (not realizable) having a pole in $z = -2$.
The probability of error is obtained by deriving the noise power at the output of the equalizer:

$$\varepsilon^2_{ZF-LE} =< \frac{N_0}{4|1 + 0.5e^{i\omega}|^2} >_{A,(-\pi,\pi)} = \frac{N_0}{4} \sum_{k=0}^{\infty} (0.5)^{2k} = \frac{N_0}{4} \frac{1}{1 - \frac{1}{4}} = \frac{N_0}{3} = \frac{0.8}{3},$$

and then using the expression for the probability of error with BPSK:

$$P_{e,ZF-LE} = Q\left(\sqrt{\frac{2E_b}{\varepsilon^2_{ZF-LE}}}\right) = Q\left(\sqrt{\frac{6}{0.8}}\right) = 3 \cdot 10^{-3}.$$

b. The pre-cursor equalizer is given by the all-pass filter

$$\frac{1}{H_0} \frac{H^*_{\max}(1/z^*)}{H_{\max}(z)} = \frac{1}{2} \frac{1 + 0.5z^{-1}}{1 + 0.5z},$$

which is clearly anti-causal stable (not realizable). The feedback filter is instead given by

$$H^*_{\max}(1/z^*) - 1 = 0.5z^{-1}.$$

The noise power at the input of the decision device is:

$$\varepsilon^2_{ZF-DFE} =< \frac{N_0}{4|1 + 0.5e^{i\omega}|^2} >_{G,(-\pi,\pi)} = \frac{N_0}{4} = 0.2,$$

so that

$$P_{e,ZF-DFE} = Q\left(\sqrt{\frac{2E_b}{\varepsilon^2_{ZF-DFE}}}\right) = Q\left(\sqrt{\frac{2}{0.2}}\right) = Q\left(\sqrt{10}\right) = 8 \cdot 10^{-4}.$$

c. The block diagram can be found in the textbook: it consists of the feedback loop moved to the transmitter side with the addition of a mod-8 operation in lieu of the decision device in the loop. The transmitted sequence is given by:

$$\begin{aligned}
x_0 &= \mathrm{mod}_8(a_0) = 3 \\
x_1 &= \mathrm{mod}_8(a_1 - 0.5x_0) = \mathrm{mod}_8(-4.5) = 3.5 \\
x_2 &= \mathrm{mod}_8(a_2 - 0.5x_1) = \mathrm{mod}_8(1.25) = 1.25 \\
x_3 &= \mathrm{mod}_8(a_3 - 0.5x_2) = \mathrm{mod}_8(0.375) = 0.375.
\end{aligned}$$

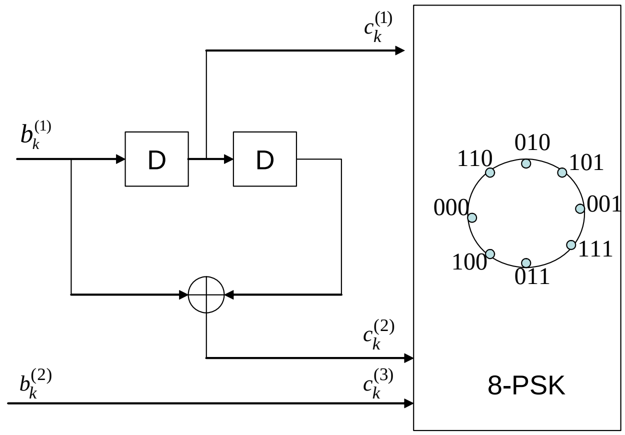

Figure 2:

P.2. Consider the trellis coded modulation scheme in the Figure below used for transmission over a standard AWGN channel.
a. Justify the choice of the 8-PSK modulation in order to have a spectral efficiency of $\rho = 2$. Moreover, find the probability of error for the corresponding uncoded system with $\rho = 2$ as a function of E/N_0.

b. Suppose that we need a coding gain of around $3dB$ as compared to the reference uncoded system above. By just looking at the encoder and at the mapping, do you think we will be able to achieve this and why?

c. Draw one section of the trellis (for the states please use the order from top to bottom 00, 10, 01, 11). Label the transitions by the input and output bits or symbol. Assuming that the all-zero sequence is transmitted (where also the uncoded bits are set to zero), evaluate the minimum distance due to the subset sequence and verify whether your prediction at the previous point is true (recall that in 8-PSK the distance between adjacent points is $2\sin\pi/8$).

Sol.: a. The uncoded system with $\rho = 2$ is 4-QAM. For this we have $d^2_{\min} = 2E$ and thus

$$\Pr[\text{symbol error}] = 2Q\left(\sqrt{\frac{E}{N_0}}\right).$$

b. The minimum distance for TCM is $d^2_{\min} = \min\{d^2_{pt},\ d^2_{subset}\}$. Since we have 4 states, by Ungerboeck's rule of thumb, the minimum distance due to the subsets d^2_{subset}, if the convolutional encoder is well designed, should guarantee a gain of around $3dB$. As for d^2_{pt}, we need $d^2_{pt} = 2 \cdot 2E$, which is exactly the value of the distance within the subsets corresponding to set partitioning level at hand (with 4 subsets and two points per subset). Overall, the given code should be able to provide a $3dB$ gain.

c. One section of the trellis is given in the Figure below. In the Figure, only the bits defining the subsets according to the given mapping are identified, parallel transitions correspond to

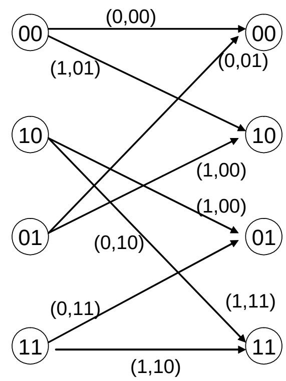

Figure 3:

the third bit being either zero or one. Assuming that the all-zero sequence is transmitted, it is easy to see that, after having resolved the parallel transitions by choosing the closest point in the subset, the closest error event is at distance

$$d^2_{subset} = 2E + (2\sin\pi/8)^2 E + 2E = 4.586E > d^2_{pt} = 4E.$$

Therefore, our conclusion at the previous point is confirmed.

P3. Consider an analog PLL with loop filter

$$L(s) = \frac{s+0.5}{s+1}.$$

a. What is the order of the PLL? Find the poles of the phase transfer function. Is the PLL stable?

b. Sketch the magnitude of the frequency response.

c. Show that there is no peaking.

d. Find the lock-in range. How can you improve this result with minimal changes to $L(s)$? Is there any drawback? Give an explicit example of your solution.

Sol.:
a. Second order. Phase transfer function:

$$\frac{\Phi(s)}{\Theta(s)} = \frac{L(s)}{L(s)+s} = \frac{s+0.5}{s^2+2s+0.5}$$

with poles: $s = -1 \pm \sqrt{2}/2 = \begin{cases} -1.7 \\ -0.3 \end{cases}$: the PLL is stable.

b. The Bode plot of the magnitude of the frequency response is easily obtained: the response is flat (0dB) up to $\log(2\pi f) = \log(0.3)$, where it starts decreasing as 20dB/decade until it reaches $\log(2\pi f) = \log 0.5$, after which it stays flat up to $\log(2\pi f) = \log(1.7)$ where it starts decreasing at 20dB/decade.

c. Follows from the inequalities

$$\left|\frac{\Phi(j\omega)}{\Theta(j\omega)}\right| = \frac{|j\omega + 0.5|}{|j\omega + 1.7|\,|j\omega + 0.3|} \leq \frac{1}{|j\omega + 1.7|} \leq 1.$$

d. Lock-in range:

$$|f_0| \leq \frac{|L(0)|}{2} = 0.25$$

This can be improved by increasing the gain of the loop filter:

$$L(s) = K_L \frac{s + 0.5}{s + 1},$$

however, this would lead to

$$\frac{\Phi(s)}{\Theta(s)} = \frac{L(s)}{L(s) + s} = \frac{K_L(s + 0.5)}{s^2 + (K_L + 2)s + 0.5K_L}.$$

Since the bandwidth is limited approximately by $1/2 \log(0.5K_L)$ (see textbook), the bandwidth is increased which creates problems in the presence of noise.