ECE 232 - Circuits and Systems II
Final

Please provide clear and complete answers. Don't forget to specify the units of measure!

1. (2 points) a. Calculate the convolution between $x(t) = e^{-3(t-1)}u(t-1)$ and a rectangle $h(t) = 0.5(u(t) - u(t-3))$.
b. Sketch the resulting function.

Sol.: We calculate the convolution between $x(t) = e^{-3t}u(t)$ and $h(t) = 0.5(u(t) - u(t-3))$. We will then delay the obtained function by 1 s. Using the usual procedure, we get the output

$$y(t) = \frac{1}{6}\left(1 - e^{-3t}\right) \text{ for } 0 \leq t \leq 3,$$
$$y(t) = \frac{1}{6}e^{-3t}(e^9 - 1) \text{ for } t \geq 3$$

and $y(t) = 0$ otherwise. After delaying by 1 s, we get

$$y(t) = \frac{1}{6}\left(1 - e^{-3(t-1)}\right) \text{ for } 1 \leq t \leq 4,$$
$$y(t) = \frac{1}{6}e^{-3(t-1)}(e^9 - 1) \text{ for } t \geq 4$$

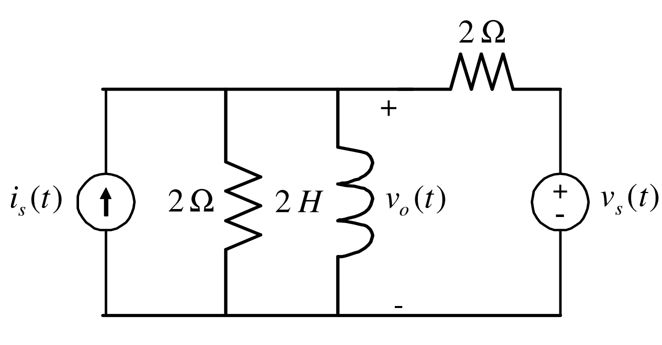
and $y(t) = 0$ otherwise.

2. (2 points) a. Draw the amplitude Bode plot for the transfer function (*Specify clearly the relevant values on the two axes*)

$$H(s) = \frac{s}{(s+100)(s+2000)}.$$

b. What type of filter is it?

Sol.: Please see the textbook on how to draw the Bode plot. It is a band-pass filter.



3. (4 points) Consider the circuit in the figure on the left. The current source $i_s(t)$ has Laplace transform $I_s(s)$ and the voltage source $v_s(t)$ has Laplace transform $V_s(s)$. Assume that the initial energy in the inductor is zero.
a. Calculate the Laplace transform $V_o(s)$ of the voltage $v_o(t)$ as a function of the Laplace transforms $I_s(s)$ and $V_s(s)$.

b. If $v_s(t) = 3\cos(2t + \pi/2)u(t)$ and $i_s(t) = 0$, find the voltage $v_o(t)$ in steady state.
c. Consider the transfer function $H(s)$ between $V_s(s)$ and $V_o(s)$. What type of filter is it? Find $H_{\max}$ and the 3dB cut-off frequency ω_c.
d. If $v_s(t) = 3\cos(2t + \pi/2)u(t)$ and $i_s(t) = 0$, find the voltage $v_o(t)$ for all $t \geq 0$ (i.e., not only in steady state). Compare with your result at point b.

Sol.: a. Using the superposition principle, we get

$$\begin{aligned} V_o(s) &= I_s(s)(1||s) + V_s(s)\frac{2||2s}{2 + 2||2s} \\ &= I_s(s)\frac{s}{1+s} + V_s(s)\frac{s}{1+2s}. \end{aligned}$$

b. The transfer function $H(s)$ between $V_s(s)$ and $V_o(s)$ is

$$H(s) = \frac{s}{1+2s}.$$

Therefore, we get

$$v_o(t) = 3|H(j2)|\cos(2t + \pi/2 + \theta(j2))u(t),$$

and, since

$$\begin{aligned} H(j2) &= \frac{j2}{1+j4} \\ &= 0.485e^{j0.245}, \end{aligned}$$

we can calculate

$$\begin{aligned} v_o(t) &= 1.455\cos(2t + \pi/2 + 0.245)u(t) \\ &= 1.455\cos(2t + 0.578\pi)u(t). \end{aligned}$$

c. We can write $H(s) = \frac{1}{2}\frac{s}{s+1/2}$. It is a high-pass filter with $H_{\max} = 1/2$ and 3-dB cut-off frequency $\omega_c = 1/2$.
d. We have

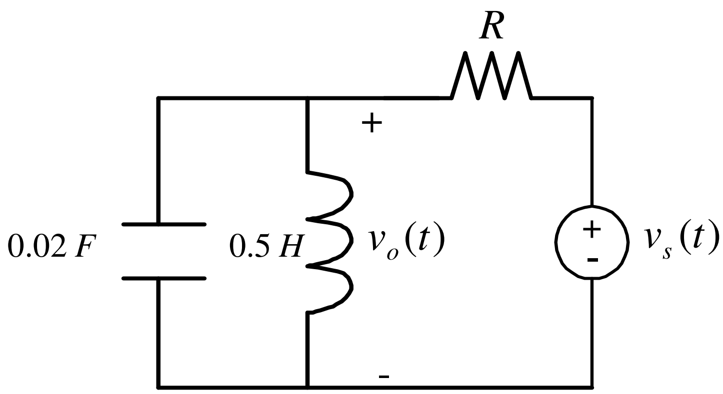
$$\begin{aligned} V_o(s) &= \frac{s}{1+2s}\frac{3s\cos(\pi/2) - 6\sin(\pi/2)}{s^2+4} \\ &= -\frac{1}{2}\frac{s}{s+1/2}\frac{6}{s^2+4} \\ &= -3\frac{s}{(s+1/2)(s^2+4)}. \end{aligned}$$

Therefore, we can now use partial fraction expansion. We have already calculated the steady-state response $1.455\cos(2t + 0.578\pi)u(t)$, so we just need to consider the transient pole $s = -1/2$. The corresponding constant is

$$\begin{aligned} K &= -3\frac{s}{s^2+4}|_{s=-1/2} \\ &= \frac{6}{17}. \end{aligned}$$

Overall we have

$$v_o(t) = 1.455\cos(2t + 0.578\pi)u(t) + \frac{6}{17}e^{-\frac{1}{2}t}u(t).$$



4. (3 points) For the circuit in the figure of the left, assume that the initial energy in the inductor and in the capacitor is zero.
a. Calculate the transfer function between the voltage source $V_s(s)$ and the output voltage $V_o(s)$ as a function of R. What type of filter is it?
b. Calculate R so that the bandwidth is 1.6 Hz.
c. For $R = 5\ \Omega$ and $v_s(t) = 5u(t)$, calculate $v_o(t)$, $t \geq 0$, in the *time domain*, that is, *without* using the Laplace transform approach.

Sol.: a. The transfer function, as seen in class, is

$$H(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2},$$

with $\omega_0 = 1/(\sqrt{LC}) = 10$ rad/s and $\alpha = 1/(2RC) = 25/R$ rad/s, so that

$$H(s) = \frac{50s/R}{s^2 + 50s/R + 100}.$$

It is a bandpass filter.
b. We have

$$\beta = 2\alpha = 50/R = 1.6 \cdot 2\pi \rightarrow R = 50/(1.6 \cdot 2\pi) \simeq 5\ \Omega.$$

c. Since $\alpha < \omega_0$, we are in the underdamped regime. The damped frequency is $\omega_d = \sqrt{100 - 25} = 8.7$ rad/s. We thus have

$$v_o(t) = 0 + B_1 e^{-5t}\cos(8.7t) + B_2 e^{-5t}\cos(8.7t),\ t \geq 0,$$

where

$$
\begin{aligned}
B_1 &= v_o(0) = 0 \\
-\alpha B_1 + \omega_d B_2 &= \frac{1}{C} i_C(0^+) \\
&= \frac{1}{0.02}\left(\frac{5}{5}\right) = 50 \\
&\rightarrow \;\; B_2 = \frac{50}{8.7} = 5.75.
\end{aligned}
$$

Overall, we get

$$v_o(t) = 5.75e^{-5t}\cos(8.7t),\;\; t \geq 0.$$