ECE 232 - Circuits and Systems II

Final

Please provide clear and complete answers. Don't forget to specify the units of measure!

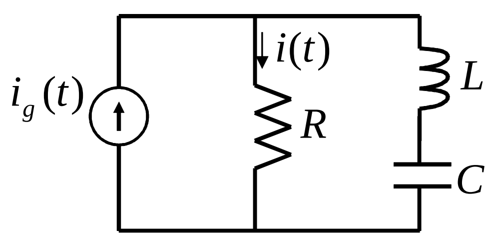
1. (**4 points**) We wish to design a high-pass filter using a capacitor of capacitance $C = 4 \times 10^{-2}$ F and a resistor of resistance R. We fix the 3-dB cut-off frequency to 4 Hz.
a. Draw the circuit by clearly specifying input and output voltages and calculate R.
b. Calculate the amplitude frequency response at 1 Hz, 4 Hz, 20 Hz.
c. Assume that the input is a constant voltage source of 2 V and that the initial voltage on the capacitor is 1 V (assume that the positive terminal is the same for both source and capacitor). Calculate the voltage $v(t)$ across the *capacitor* for all $t \geq 0$ using *time domain* analysis.
d. Confirm the result at the previous point by calculating $v(t)$ for all $t \geq 0$, when the input is a constant voltage source of 2 V and the initial voltage on the capacitor is 1 V, via partial fraction expansion.

2. (**2 points**) Calculate the convolution between $x(t) = t(u(t) - u(t-1))$ and $h(t) = (1-t)(u(t) - u(t-1))$ (don't forget to check for the continuity of the function).

3. (**2 points**) Calculate the Laplace transform of

$$(t-4)\sin(3t-12)u(t-4)$$

by detailing all the steps.



4. (**4 points**) Consider the circuit in the figure. The input is $i_g(t)$ and the output is $i(t)$.
a. What kind of filter is it? Argue your response through considerations on the impedances in the circuit.
b. Calculate the transfer function $H(s)$.
c. Assume that $L = 1$ H, $C = 1$ F and $R = 1$ Ω. Calculate and plot poles and zeros.
d. If the input is $i_g(t) = 6\cos(t + \pi/4)$ calculate the steady-state output $i(t)$.

5. (**1 point**) Draw the Bode plot for the transfer function $H(s) = \frac{s+0.1}{s+1000}$.

Solutions:
1. a. Please see class notes or text. The input is a voltage source and the output is the voltage across the resistor. We also have $\omega_c = 1/\tau$ and hence

$$R = 1/(\omega_c C) = 1/(2\pi 4 \cdot 4) \times 10^2$$
$$= 0.995\ \Omega.$$

b. We have

$$H(s) = \frac{R}{R + 1/(sC)} = \frac{s}{s + 1/\tau}$$

and thus we can calculate $|H(j2\pi)| = 0.24$, $|H(j2\pi4)| = 1/\sqrt{2}$, and $|H(j2\pi20)| = 0.98$.
c. This is a step response problem. We get

$$\begin{aligned} v(t) &= v(\infty) + (v(0) - v(\infty))e^{-t/\tau} \\ &= 2 + (1-2)e^{-2\pi4t} \\ &= 2 - e^{-2\pi4t} \end{aligned}$$

for $t \geq 0$.
d. In the Laplace domain, we use the equivalent series for the capacitor, which adds a voltage source of $1/s$, to get

$$\begin{aligned} V(s) &= \left(\frac{2}{s} - \frac{1}{s}\right)\frac{2\pi4}{s + 2\pi4} + \frac{1}{s} \\ &= \frac{2\pi4}{s(s+2\pi4)} + \frac{1}{s}, \end{aligned}$$

since the transfer function between the voltage source and the voltage on the capacitor is $\frac{2\pi4}{s+2\pi4}$ and the second term is due to the equivalent series representation for the capacitor. Using partial fraction expansion, we write:

$$K_1 = 1$$

and

$$K_2 = \frac{2\pi4}{-2\pi4} = -1.$$

From which we get

$$v(t) = 1 - e^{-2\pi4t} + 1 = 2 - e^{-2\pi4t},$$

as desired.

2. For $0 \leq t \leq 1$, we have

$$\begin{aligned} y(t) &= \int_0^t (1-\tau)(t-\tau)d\tau \\ &= \int_0^t (t - \tau - \tau t + \tau^2)d\tau \\ &= t^2 - \frac{1}{2}(1+t)\tau^2|_0^t + \frac{1}{3}\tau^3|_0^t \\ &= t^2 - \frac{1}{2}(1+t)t^2 + \frac{1}{3}t^3 = \\ &= \frac{1}{2}t^2 - \frac{1}{6}t^3. \end{aligned}$$

For $1 \le t \le 2$, we have

$$
\begin{aligned}
y(t) &= \int_{t-1}^{1} (1-\tau)(t-\tau)d\tau \\
&= t(1-t+1) - \frac{1}{2}(1+t)\tau^2|_{t-1}^{1} + \frac{1}{3}\tau^3|_{t-1}^{1} \\
&= t(2-t) - \frac{1}{2}(1+t)(1-(t-1)^2) + \frac{1}{3}(1-(t-1)^3).
\end{aligned}
$$

And we also have $y(t) = 0$ for $t \le 0$ and $t \ge 2$. Continuity at $t = 0, 1, 2$ is easy to prove.

3. Since

$$
\begin{aligned}
\mathcal{L}(t\sin(3t)u(t)) &= -\frac{d}{ds}\frac{3}{s^2+9} \\
&= \frac{6s}{(s^2+9)^2}
\end{aligned}
$$

then

$$
\mathcal{L}((t-4)\sin(3(t-4))u(t-4)) = e^{-4s}\frac{6s}{(s^2+9)^2}.
$$

4. a. It is a band-reject filter since it can be seen that $I(j\omega) = I_g(j\omega)$ for both high and low frequencies: the system performs current division and the impedance of the series LC branch goes to infinity in both cases. Moreover, similar to the discussion in class, it can be easily seen that $I(j\omega_c) = 0$ for $\omega_c = 1/\sqrt{LC}$ since the impedance of the LC branch satisfies $j\omega_c L + 1/(j\omega_c C) = 0$.
b. We have

$$
\begin{aligned}
H(s) &= \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}} \\
&= \frac{s^2 + \omega_0^2}{s^2 + 2\alpha s + \omega_0^2}
\end{aligned}
$$

with $\omega_0^2 = 1/(LC)$ and $\alpha = R/(2L)$.
c. With the given values, we can calculate $\omega_0^2 = 1$ and $\alpha = 1/2$. The poles are

$$
\begin{aligned}
s &= -\alpha + j\sqrt{\omega_0^2 - \alpha^2} = -\alpha + j\frac{\sqrt{3}}{2} \\
s &= -\alpha + j\sqrt{\omega_0^2 - \alpha^2} = -\alpha - j\frac{\sqrt{3}}{2}.
\end{aligned}
$$

The zeros are $\pm j1$.
d. The output is easily seen to be zero in steady state since $H(j1) = 0$.

5. We write

$$\begin{aligned}|H(j\omega)| &= \frac{0.1|1+j\frac{\omega}{0.1}|}{1000|1+j\frac{\omega}{1000}|} \\ &= 10^{-4}\frac{|1+j\frac{\omega}{0.1}|}{|1+j\frac{\omega}{1000}|}.\end{aligned}$$

Using the usual rules, the plot is constant and equal to -80 dB up to $\omega = 0.1$ rad/s. It then goes up by 20 dB/decade thus reaching 0 dB at $\omega = 1000$ rad/s. It then stays flat at this value.