Random signl analysis I (ECE673)
Solution assignment 7

1) (i) Given the independent random variables $X_1 \sim Ber(0.3)$ and $X_2 \sim Ber(0.6)$, evaluate the mean vector and covariance matrix of the transformed random vector

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 + X_2 \\ X_2 \end{bmatrix}.$$

(ii) Write a MATLAB program that estimates mean vector and covariance matrix of $\mathbf{Y}$ from Monte Carlo simulations. Compare the simulation results with the analysis performed at point (i). Please include the MATLAB code and numerical outcome.

Solution: (i) The mean vector reads

$$E[\mathbf{Y}] = \begin{bmatrix} E[X_1] + [X_2] \\ E[X_2] \end{bmatrix} = \begin{bmatrix} 0.3 + 0.6 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.6 \end{bmatrix},$$

whereas the covariance matrix can be evaluated as follows:

$$\begin{aligned} var(Y_1) &= var(X_1) + var(X_2) = 0.3 \cdot 0.7 + 0.6 \cdot 0.4 = 0.45 \\ var(Y_2) &= var(X_2) = 0.6 \cdot 0.4 = 0.24 \\ cov(Y_1, Y_2) &= E[(Y_1 - E[Y_1])(Y_2 - E[Y_2])] = E[(X_1 - E[X_1] + X_2 - E[X_2])(X_2 - E[X_2])] = \\ &= E[(X_1 - E[X_1])(X_2 - E[X_2])] + E[(X_2 - E[X_2])(X_2 - E[X_2])] = \\ &= cov(X_1, X_2) + var(X_2) = 0 + 0.24 = 0.24, \end{aligned}$$

therefore

$$\mathbf{C}_{\mathbf{Y}} = \begin{bmatrix} var(Y_1) & cov(Y_1, Y_2) \\ cov(Y_1, Y_2) & var(Y_2) \end{bmatrix} = \begin{bmatrix} 0.45 & 0.24 \\ 0.24 & 0.24 \end{bmatrix}.$$

(ii) A possible MATLAB program to estiamate mean and covariance is as follows:

```
N=100000; %Monte Carlo iterations
p1=0.3; %Bernoulli parameters p
p2=0.6;
Y=zeros(2,N);
for i=1:N
U=rand(2,1); %uniform variables for the inverse probability integral method
 if (U(1)>1-p1) X(1)=1; else X(1)=0; end
if (U(2)>1-p2) X(2)=1; else X(2)=0; end
Y(1,i)=X(1)+X(2);
Y(2,i)=X(2);
end
EY=zeros(2,1);
for j=1:N
EY=EY+Y(:,j);
end
EY=1/N*EY
CY=zeros(2);
for j=1:N
```

CY=CY+(Y(:,j)-EY)*(Y(:,j)-EY)';
end
CY=1/N*CY

The numerical outcome for the given realizations are:
EY =
0.9040
0.6026
CY =
0.4488 0.2388
0.2388 0.2395.

2) Two lightbulbs have times to failure described by random variables X and Y (measured in months) respectively with joint PDF

$$p_{X,Y}(x,y) = 10^{-4}\exp[-0.01(x+y)]u(x)u(y),$$

where it is recalled that $u(x)$ is the step function ($u(x) = 1$ for $x \geq 0$ and $u(x) = 0$ for $x < 0$). (i) Evaluate the marginal PDFs of X and Y. Are they independent? (ii) Calculate the mean vector $E[\mathbf{Z}]$ with $\mathbf{Z} = \begin{bmatrix} X \\ Y \end{bmatrix}$ and the covariance matrix $\mathbf{C_Z}$. (iii) Evaluate the probability that both lightbulbs fail before 50 months.
Solution: (i) The marginal PDF for X is calculated as

$$\begin{aligned} p_X(x) &= \int_{-\infty}^{+\infty} p_{X,Y}(x,y)dy = \int_{-\infty}^{+\infty} 0.0001\exp[-0.01(x+y)]u(x)u(y)dy = \\ &= 0.01\exp[-0.01x]u(x)\int_{-\infty}^{+\infty} 0.01\exp[-0.01y]u(y)dy = \\ &= 0.01\exp[-0.01x]u(x), \end{aligned}$$

where the last equality follows from the fact that $\lambda\exp(-\lambda y)$ inside the integral ($\lambda = 0.01$) is an exponential PDF. It follows that $X \sim \exp(0.01)$ and, similiarly $Y \sim \exp(0.01)$. Since $p_{X,Y}(x,y) = p_X(y)p_Y(y)$, X and Y are independent.
(ii) The mean vector reads

$$E[\mathbf{Z}] = \begin{bmatrix} E[X] \\ E[Y] \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

and the coviariance matrix is diagonal since the variables are independent and

$$\mathbf{C}_Z = \begin{bmatrix} var(X) & 0 \\ 0 & var(Y) \end{bmatrix} = \begin{bmatrix} 10000 & 0 \\ 0 & 10000 \end{bmatrix}.$$

(iii) The requested probability is

$$\begin{aligned} P[X &\leq 50, Y \leq 50] = \int_{-\infty}^{50}\int_{-\infty}^{50} p_{X,Y}(x,y)dxdy = \int_{-\infty}^{50} p_X(x)dx \int_{-\infty}^{50} p_Y(y)dy = \\ &= F_X(50)F_Y(50) = (1-e^{-0.01\cdot 50})^2 = 0.15. \end{aligned}$$

3) The amplitudes of two voice signals (X_1 and X_2) are modelled as bivariate jointly Gaussian variables (measured in Volt) with zero mean , variance 1 and correlation coefficient $\rho = 0.8$. (i) Plot the joint PDF using MATLAB. In particular, show both the tri-dimensional plot (using the command *mesh*) and the countour lines (using the command *contour*). (ii) Evaluate the probability that the amplitude of the second signal is larger than the first by 1 Volt ($P[X_2 - X_1 > 1]$).

Solution: (i) In order to plot the standard bivariate Gaussian variables X_1 and X_2 we can use the following code

```
x1=
x2=
[X1,X2]=meshgrid(x1,x2);
P=1/
mesh(X1,X2,P); %for the tri-dimensional plot
figure; %creates a new figure
```

(ii) The requested probability can be written as

$$P[X_2 - X_1 > 1],$$

but $X_2 - X_1$ is known to be a Gaussian variable (linear combination of random variables), whose mean and variance can be determined as follows

$$\begin{aligned} E[X_2 - X_1] &= 0 - 0 = 0 \\ var(X_2 - X_1) &= var(X_2) + var(-X_1) + 2cov(X_2, -X_1) = \\ &= 1 + 1 - 2\rho = 0.4. \end{aligned}$$

Therefore, we can write

$$X_2 - X_1 = \sqrt{0.4}Z,$$

where $Z \sim \mathcal{N}(0, 1)$. It follows that

$$P[X_2 - X_1 > 1] = P[Z > 1/\sqrt{0.4}] = Q(1/\sqrt{0.4}) = 0.0569,$$

where the $Q(x)$ function has been evaluated with MATLAB using the command 1/2*erfc(x/sqrt(2)).

4) The voice signals defined in the previous point are passed through an amplifier that accepts a maximum signal level of 2.5 (i.e., the dynamic range of the amplifier is $[-2.5, 2.5]$). Knowing that $X_2 = 2$, what is the probability that X_1 is outside the dynamic range of the amplifier?

Solution: Since the event $X_2 = 2$ is known, we need to consider the conditional PDF $p_{X_1|X_2}(x_1|2)$. We known that for standard bivariate Gaussian variables, the conditional PDF $p_{X_1|X_2}(x_1|x_2)$ is Gaussian with mean ρx_2 and variance $1 - \rho^2$, therefore we have

$$X_1|(X_2 = 2) \sim \mathcal{N}(2 \cdot 0.8, 1 - (0.8)^2) \sim \mathcal{N}(1.6, 0.36),$$

which implies

$$\{X_1|(X_2 = 2)\} = 1.6 + \sqrt{0.36}Z,$$

where $Z \sim \mathcal{N}(0,1)$. Therefore the requested probability reads

$$\begin{aligned}
P[|X_1| &> 2.5|X_2 = 2] = P[X_1 > 2.5|X_2 = 2] + P[X_1 < -2.5|X_2 = 2] = \\
&= P[1.6 + \sqrt{0.36}Z > 2.5] + P[1.6 + \sqrt{0.36}Z < -2.5] = \\
&= P[Z > 0.9/\sqrt{0.36}] + P[Z < -4.1/\sqrt{0.36}] = \\
&= P[Z > 1.5] + P[Z < -6.8] = Q(1.5) + 1 - Q(-6.8) = \\
&= Q(1.5) + Q(6.8) = 0.067 + 5 \cdot 10^{-12} = 0.067.
\end{aligned}$$