ECE 232 - Circuits and Systems II
Midterm

Please provide clear and complete answers. Don't forget to specify the units of measure!

Q1. (1 point) Plot the function $v(t) = 4e^{-2000t}\sin(2\pi \cdot 1000t)$ $[V]$ for $0 \leq t \leq 2.5$ ms.

Sol.: The time constant of the exponential term is $1/2000 = 0.5$ ms and the period of the sinusoid is $1/1000 = 1$ ms.

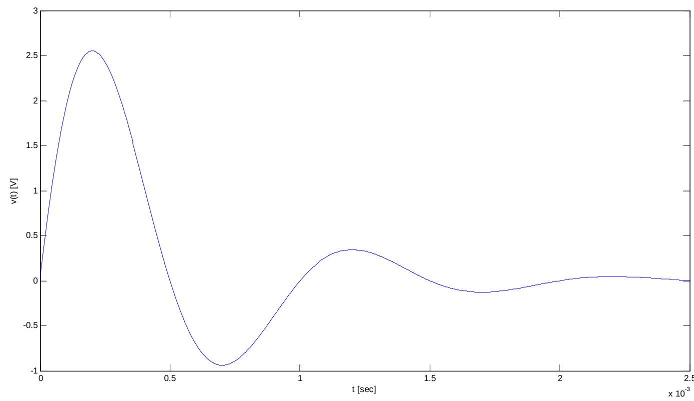

Figure 1:

Q2. (1 point) Calculate the Laplace transform of

$$f(t) = 2 \cdot (t-1) \cdot e^{-20(t-1)}u(t-1).$$

(Hint: Start with the Laplace transform of $te^{-20t}u(t)$)

Sol: We know that

$$\mathcal{L}(e^{-20t}u(t)) = \frac{1}{s+20},$$

so that

$$\mathcal{L}(2te^{-20t}u(t)) = 2\frac{d}{ds}\left(\frac{1}{s+20}\right) = \frac{2}{(s+20)^2}.$$

We finally have

$$\mathcal{L}(2 \cdot (t-1) \cdot e^{-20(t-1)}u(t-1)) = e^{-s}\frac{2}{(s+20)^2}.$$

P1. (4 points) Consider the circuit in the figure below. Assume that for $t < 0$ switch 1 is closed and switch 2 open and that the circuit has been in this configuration for a long time.
a. Calculate $v_c(t)$ for $0 \leq t < 8ms$. In this interval of time, switch 1 and 2 are open.
b. Calculate $v_c(t)$ for $t \geq 8ms$. In this interval of time, switch 1 is open and 2 closed.

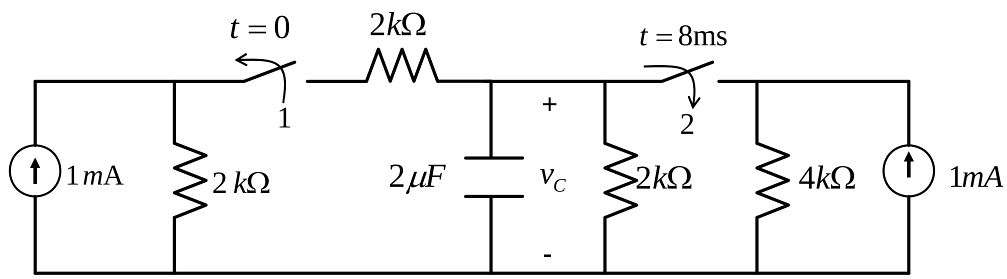

Figure 2:

c. Calculate the initial energy available in the capacitor and the fraction of such energy dissipated by the resistors in the interval $0 \le t < 8ms$.
d. Double check your result at the previous point by calculating the energy remaining on the capacitor at time $t = 8ms$. Explain.

Sol.: a. We have

$$v_c(0) = 2 \cdot 10^3 \cdot \frac{2}{6} 10^{-3} = \frac{2}{3} \text{ V}.$$

Moreover, for $0 \le t < 8ms$, the circuit is RC with $\tau = RC = 2 \cdot 10^3 \cdot 2 \cdot 10^{-6} = 4$ ms. Therefore, we have

$$v_c(t) = \frac{2}{3} e^{-250t}.$$

b. For $t \ge 8ms$, the circuit is RC with $R = 2 \cdot 4/6 = 4/3\ k\Omega$, so that $\tau = 8/3ms$, and a source of 1 mA. Moreover, we can calculate

$$v_c(8 \cdot 10^{-3}) = \frac{2}{3} e^{-2} = 0.09 \text{ V}$$

and

$$v_c(\infty) = \frac{4}{3} \cdot 10^3 \cdot 10^{-3} = \frac{4}{3} \text{ V}.$$

Finally, we obtain

$$\begin{aligned} v_c(t) &= \frac{4}{3} + \left(0.09 - \frac{4}{3}\right) e^{-\frac{3000}{8}(t - 8 \cdot 10^{-3})} \\ &= \frac{4}{3} - 1.24 e^{-375(t - 8 \cdot 10^{-3})}. \end{aligned}$$

c. Initial energy available in the capacitor:

$$\frac{1}{2} C v_c(0)^2 = \frac{1}{2} 2 \cdot 10^{-6} \cdot \frac{4}{9} = \frac{4}{9} \cdot 10^{-6} \text{ J}.$$

Energy dissipated by the $2k\Omega$ resistor connected to the capacitor in the interval $0 \le t < 8ms$

$$\begin{aligned} \int_0^{8 \cdot 10^{-3}} \frac{v_c(t)^2}{2000} dt &= \frac{2}{9000} \int_0^{8 \cdot 10^{-3}} e^{-500t} dt \\ &= \frac{2}{9000 \cdot (-500)} (e^{-500 \cdot 8 \cdot 10^{-3}} - 1) \\ &= 4.36 \cdot 10^{-7}, \end{aligned}$$

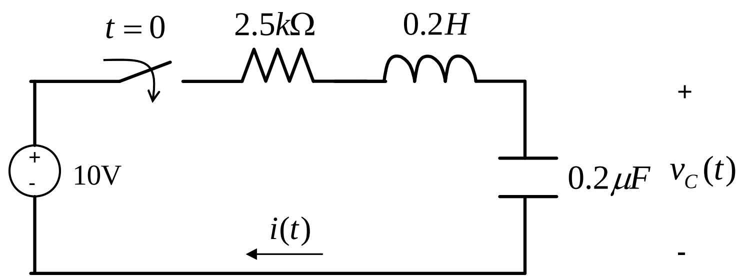

Figure 3:

which is a fraction

$$\frac{4.36 \cdot 10^{-7}}{\frac{4}{9} \cdot 10^{-6}} \cdot 100 = 98.1\%.$$

d. The energy remaining in the capacitor is

$$\frac{1}{2} C v_c(\infty)^2,$$

which can be seen to be around 1.9% of the initial energy.

P2. (4 points) In the circuit in the figure below, no energy is initially stored in inductor and capacitor.
a. Calculate the roots of the characteristic equation and determine the regime.
b. Find $v_C(0)$ and $\frac{dv_C(0^+)}{dt}$.
c. Find $v_C(t)$ for $t \geq 0$.
d. Find $i(t)$ for $t \geq 0$.

Sol.: a. We have

$$\begin{aligned} \alpha &= \frac{R}{2L} = \frac{2.5 \cdot 10^3}{2 \cdot 0.2} = \frac{2.5 \cdot 10^3}{0.4} = 6.25 \cdot 10^3 \text{ rad/s} \\ \omega_0 &= \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2 \cdot 0.2 \cdot 10^{-6}}} = \frac{1}{\sqrt{4 \cdot 10^{-8}}} = 5 \cdot 10^3 \text{ rad/s} \end{aligned}$$

so we are in the overdamped regime. The roots are

$$\begin{aligned} s_1 &= -6.25 \cdot 10^3 + 10^3 \sqrt{6.25^2 - 5^2} \\ &= -6.25 \cdot 10^3 + 3.75 \cdot 10^3 = -2500 \\ s_2 &= -6250 - 3750 = -10000. \end{aligned}$$

b.

$$\begin{aligned} v_C(0) &= 0 \text{ V} \\ \frac{dv_C(0^+)}{dt} &= \frac{1}{C} i_C(0^+) = 0 \text{ V/s}, \end{aligned}$$

since the initially stored energy is zero.
c. From the general formula for the overdamped regime, we have

$$v_C(t) = v_C(\infty) + A_1 e^{-2500t} + A_2 e^{-10000t} \text{ V},\ t \geq 0$$

where

$$v_C(\infty) = 10 \text{ V},$$

and imposing the initial conditions, we get

$$\begin{aligned} 10 + A_1 + A_2 &= 0 \\ -2500A_1 - 10000A_2 &= 0, \end{aligned}$$

and

$$\begin{aligned} A_1 &= -40/3 \\ A_2 &= 10/3, \end{aligned}$$

so that

$$v_C(t) = 10 - \frac{40}{3}e^{-2500t} + \frac{10}{3}e^{-10000t} \text{ V},\ t \geq 0.$$

d. From the general formula for the underdamped regime, we have

$$i(t) = A_1 e^{-2500t} + A_2 e^{-10000t} \text{ A},\ t \geq 0$$

and imposing the initial conditions

$$\begin{aligned} A_1 + A_2 &= i(0) = 0 \\ -2500B_1 - 10000A_2 &= \frac{di(0^+)}{dt} = \frac{1}{L}v_L(0^+) = \frac{1}{0.2}10 = 50, \end{aligned}$$

and

$$\begin{aligned} A_1 &= 2/3 \cdot 10^{-3} \\ A_2 &= -2/3 \cdot 10^{-3}, \end{aligned}$$

so that

$$i(t) = \frac{2}{3}e^{-2500t} - \frac{2}{3}e^{-10000t}\ mA,\ t \geq 0.$$