ECE 232 - Circuits and Systems II
Midterm

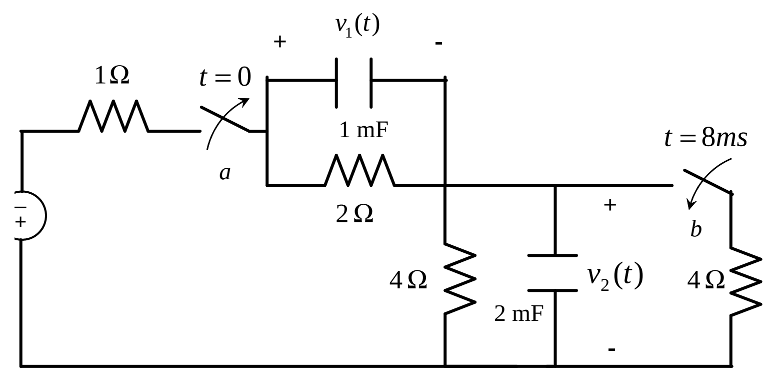
Please provide clear and complete answers. Don't forget to specify the units of measure!

1. (2 point) Calculate the Laplace transform of

$$f(t) = 2e^{-20(t-1)}\cos(10(t-1))u(t-1)$$

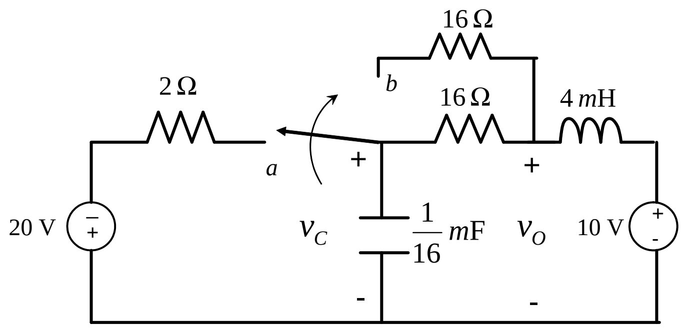
starting from the Laplace transform of $\cos(10t)u(t)$ and then applying some of the properties of the Laplace transform (which ones? please detail all the steps).

2. (2 point) Consider the parallel of an inductor $L = 1\ H$ and a capactior $C = 1\ F$. Assume that the initial energy in the capacitor is zero, while the energy in the inductor is $0.5\ J$ (choose any direction you want for the current). Find the voltage $v(t)$ for $t \geq 0$ (Hint: It is a parallel RLC circuit with infinite resistance).



3. (4 points) Consider the circuit in figure 1. Switch a opens at time $t = 0$, after having being closed for a long time, while switch b closes at time $t = 8\ ms$.
a. Calculate $v_1(0)$ and $v_2(0)$.
b. Calculate $v_1(t)$ and $v_2(t)$ for $0 \leq t \leq 8\ ms$
c. Calculate $v_1(t)$ and $v_2(t)$ for $t \geq 8\ ms$.
d. Which percentage of the initial energy in the 1 mF capacitor is dissipated in the interval $0 \leq t \leq 8\ ms$?

4. (4 points) In the circuit illustrated in figure 2 the switch stays in position a for a long time and then moves to position b at time $t = 0$.
a. Calculate $v_C(0^-)$, $v_C(0^+)$ and $\frac{dv_C(0^+)}{dt}$, where $v_C(t)$ is the voltage across the capacitor as shown in the figure.
b. Calculate $v_C(t)$ for $t \geq 0$.
c. Calculate $v_O(0^-)$, $v_O(0^+)$ and $\frac{dv_O(0^+)}{dt}$, where $v_O(t)$ is the voltage over the capacitor and the resistors in parallel as shown in the figure.
d. Calculate $v_O(t)$ for $t \geq 0^+$.



Sol.:

1. We know that

$$\mathcal{L}(\cos(10t)u(t)) = \frac{s}{s^2+100}.$$

Now, using the multiplication by a constant and multiplication by an exponential properties, we get

$$\begin{aligned}\mathcal{L}(2e^{-20t}\cos(10t)u(t)) &= 2\frac{s+20}{(s+20)^2+100} \\ &= 2\frac{s+20}{s^2+40s+500}.\end{aligned}$$

We conclude by using the delay property

$$\mathcal{L}(2e^{-20(t-1)}\cos(10(t-1))u(t-1)) = 2e^{-s}\frac{s+20}{s^2+40s+500}.$$

2. Since R is infinity, we have $\alpha = \frac{1}{2RC} = 0$. Moreover, since $\omega_0 = 1$ rad/s, we are in the underdamped regime and

$$v(t) = B_1 \cos t + B_2 \sin t.$$

Imposing the initial conditions, we have

$$\begin{aligned} v(0) &= 0 \rightarrow B_1 = 0 \\ \frac{dv(0^+)}{dt} &= \frac{1}{C} i(0^+) = 1 \rightarrow B_2 = 1. \end{aligned}$$

In fact, the initial current follows from $\frac{1}{2}Li(0^+)^2 = \frac{1}{2}$. In the second condition, we have taken the direction of the current so that it flows from the + to the - terminal on the capacitor. Overall, we have

$$v(t) = \sin t \text{ for } t \geq 0.$$

3.

a. From the voltage division rule, we have

$$\begin{aligned} v_1(0) &= -7\frac{2}{1+2+4} = -2\ V \\ \text{and } v_2(0) &= -7\frac{4}{1+2+4} = -4\ V. \end{aligned}$$

b. For $0 \leq t \leq 8\ ms$, we have to separate RC circuits. In fact, from Kirchhoff current law, the current that flows over the 1 mF capacitor is the same as over the 2 Ω resistor and similarly for the 2 mF capacitor and the 4 Ω resistor. The two time constants are

$$\begin{aligned} \tau_1 &= 2\ ms \\ \text{and } \tau_2 &= 8\ ms, \end{aligned}$$

so that we have

$$\begin{aligned} v_1(t) &= -2e^{-500t}\ \text{V} \\ \text{and } v_2(t) &= -4e^{-125t}\ \text{V} \end{aligned}$$

for $0 \leq t \leq 8\ ms$.

c. In the interval $t \geq 8\ ms$, the first RC circuit does not change so that

$$v_1(t) = -2e^{-500t}\ \text{V}$$

also for $t \geq 8\ ms$.
Instead, for the second RC circuit, by continuity of the voltage over capacitors, we have

$$v_2(8ms) = -4e^{-1} = -1.47\ \text{V}$$

and the time constant is

$$\tau_2 = 4\ ms$$

since the equivalent resistance is $4||4\Omega = 2\Omega$. We can conclude that

$$v_2(t) = -1.47e^{-250(t-8\cdot 10^{-3})}\ V$$

for $t \geq 8\ ms$.
d. The initial energy is

$$E_1(0) = \frac{1}{2}10^{-3}(-2)^2 = 2\ mJ$$

while the energy at $t = 8\ ms$ is given almost zero since $8\ ms = 4\tau_1$. Therefore, the percentage dissipated is very close to 100%. Precisely, we have

$$E_1(8ms) = \frac{1}{2}10^{-3}(-2e^{-500\cdot 8\cdot 10^{-3}})^2 = 0.67\ \mu J$$

so that the percentage dissipated is

$$\frac{E_1(0) - E_1(8ms)}{E_1(0)} = 99.9\%.$$

4.

a. For $t = 0^-$, we can evaluate the current flowing in the inductor, with direction from the 16Ω to the $2\ \Omega$ resistor as

$$i_L(0^-) = \frac{30}{18} = \frac{5}{3} = 1.67\ A,$$

since we have an equivalent source of $30\ V$. It follows that

$$\begin{aligned} v_C(0^-) &= v_C(0^+) = v_C(0) = -20 + 2 \cdot 1.67 \\ &= -16.67\ V \end{aligned}$$

(you can also calculate it as $10 - 16 \cdot 1.67 = -16.72\ V$).
Moreover, after the switch has moved to position b, we can calculate

$$\frac{dv_C(0^+)}{dt} = \frac{1}{C} i_L(0^+) = 16000 \cdot 1.67 = 26,720\ A/s.$$

b. We have a series RLC circuit with

$$\alpha = \frac{R}{2L} = \frac{8}{2 \cdot 4 \cdot 10^{-3}} = 1000 \text{ rad/s}$$

and

$$\omega_0 = \frac{1}{\sqrt{4 \cdot 10^{-3} \cdot \frac{1}{16} \cdot 10^{-3}}} = 2000 \text{ rad/s},$$

so that $\omega_d = \sqrt{2000^2 - 1000^2} = 1732.1$ rad/s. We are then in the underdamped regime. It follows that

$$v_C(t) = B_1 e^{-1000t} \cos(1732.1t) + B_2 e^{-1000t} \sin(1732.1t) + 10\ V.$$

Imposing the initial conditions

$$\begin{aligned} B_1 + 10 &= -16.67 \\ -1000 B_1 + 1732.1 B_2 &= 26720 \\ &\rightarrow B_2 = 0.03 \end{aligned}$$

we get

$$v_C(t) = -26.67 e^{-1000t} \cos(1732.1t) + 0.03 e^{-1000t} \sin(1732.1t).$$

c. For $t = 0^-$, we have

$$\begin{aligned} v_O(0^-) &= v_C(0^-) + 8 \cdot 1.67 \\ &= -16.67 + 8 \cdot 1.67 \\ &= -3.31\ V \end{aligned}$$

whereas for $t = 0^+$, we have

$$\begin{aligned} v_O(0^+) &= v_C(0^+) + 16 \cdot 1.67 \\ &= -16.67 + 16 \cdot 1.67 \\ &= 10.05\ V. \end{aligned}$$

To obtain $\frac{dv_O(0^+)}{dt}$, we can differentiate the expression for $v_O(t)$ obtained at the following point.

d. For $t \geq 0^+$, we calculate

$$v_O(t) = v_C(t) + 8 \cdot i(t),$$

where $i(t)$ is the current flowing in the RLC circuit (in the direction from the inductor to the capacitor). This can be calculated for instance by differentiating the voltage $v_C(t)$. We do not further detail this point here.