ECE 232 - Circuits and Systems II
Midterm

Please provide clear and complete answers. Don't forget to specify the units of measure!

1. (2 points) Consider an RL circuit with with $R = 1\ \Omega$ and $L = 2\ H$. The initial energy stored in the inductor is $4\ J$. Find an equivalent circuit in the s-domain, and use this circuit to evaluate the Laplace transform $I(s)$ of the current $i(t)$ for $t \geq 0$. From this result, obtain the current $i(t)$ for $t \geq 0$.

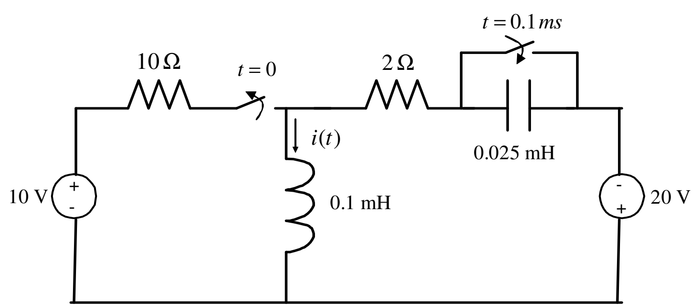
2. (2 points) Calculate the Laplace transform of

$$f(t) = 2(t-1)e^{-20(t-1)}u(t-1)$$

and of

$$f(t) = 4e^{-3t}\sin(10t)u(t)$$

using some of the properties of the Laplace transform (which ones? Please detail all the steps). Calculate and illustrate poles and zeros for both Laplace transforms.



3. (4 points) Consider the circuit in the figure, where the first switch opens at time $t = 0$ after a long time, and the second switch closes at time $t = 0.1\ ms$. We refer to the current in the inductor as $i(t)$ (see figure).
a. Calculate $i(0)$ and $di(0^+)/dt$.
b. Calculate $i(t)$ for $0 \leq t \leq 0.1\ ms$.
c. Calculate $i(t)$ for $t \geq 0.1\ ms$.

4. (2 points) a. Calculate the impedance of the parallel of a resistor of $1\ \Omega$, an inductor of $1\ H$ and a capacitor of $1\ F$ (Hint: Obtain first the equivalent circuit in the s-domain. You can assume that no initial energy is stored).
b. Evaluate poles and zeros of the impedance.
c. How would you change the the value of the resistance so as to obtain steady-state poles?

5. (2 points) Plot the following function

$$\begin{aligned} f(t) &= tu(t) + (2-2t)u(t-1) + (-3+t)u(t-3) \\ &\quad +(e^{-(t-4)} - 1)u(t-4). \end{aligned}$$

Sol.:

1. From the initial energy, we obtain

$$i(0) = \sqrt{\frac{2}{L}4} = 2\ A.$$

Please see the book on how to obtain an equivalent circuit. Using the parallel equivalent for the inductor, we obtain a voltage source of $Li(0) = 4\ V \cdot s$. We thus obtain

$$I(s) = \frac{Li(0)}{R + Ls} = \frac{i(0)}{s + \frac{1}{\tau}} = \frac{2}{s + 1/2}$$

and

$$i(t) = 2e^{-t/2}u(t).$$

2. We obtain

$$\begin{aligned} F(s) &= 2e^{-s}\mathcal{L}\{te^{-20t}u(t)\} \\ &= 2e^{-s}\left(-\frac{d}{ds}\mathcal{L}\{e^{-20t}u(t)\}\right) \\ &= \frac{2e^{-s}}{(s+20)^2}, \end{aligned}$$

by using the properties of multiplication by a constant, translation in time and multiplication by t in this order. There are no zeros and two poles in -20.
For the second function, we get

$$\begin{aligned} F(s) &= 4\frac{10}{(s+3)^2 + 100} \\ &= \frac{40}{(s+3)^2 + 100}, \end{aligned}$$

where we have used the properties of multiplication by a constant and multiplication by an exponential function. There are no zeros and the poles are in $s = -3 \pm j10$.
3.
a. Observing the system at time $t = 0^-$, we have

$$i(0^-) = i(0) = 1\ A$$

and

$$v_c(0^-) = v_c(0) = -20\ V,$$

where we chose as positive terminal the one on the side of the resistor. We thus get

$$\begin{aligned} \frac{di(0^+)}{dt} &= \frac{1}{L}v_L(0^+) = \frac{1}{L}\left(-20 + 20 - 2i(0)\right) \\ &= -2 \cdot 10^4\ A/s. \end{aligned}$$

b. For $0 \le t \le 0.1\ ms$, we have a series RLC with

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2 \cdot 10^4\ rad/s$$

and

$$\alpha = \frac{R}{2L} = 10^4 \ rad/s.$$

Therefore, we are in the underdamped regime, with

$$\begin{aligned} s_1 &= -10^4 + j1.73 \cdot 10^4 \\ s_2 &= -10^4 - j1.73 \cdot 10^4 \end{aligned}$$

so that

$$i(t) = i(\infty) + e^{-10^4 t}(B_1 \cos(1.73 \cdot 10^4 t) + B_2 \sin(1.73 \cdot 10^4 t)).$$

We further obtain from the initial conditions

$$\begin{aligned} B_1 &= 1 \\ -10^4 B_1 + 1.73 \cdot 10^4 B_2 &= -2 \cdot 10^4, \end{aligned}$$

which leads to $B_2 = (-2+1)/1.73 = -0.58$. The final value is $i(\infty) = 0 \ A$. Overall, we have

$$i(t) = e^{-10^4 t}(\cos(1.73 \cdot 10^4 t) - 0.58 \sin(1.73 \cdot 10^4 t)) \ A.$$

c. For $t \geq 0.1 \ ms$, the system becomes an RL circuit. Moreover, we have

$$\begin{aligned} i(0.1ms) &= e^{-1}(\cos(1.73) - 0.58 \sin(1.73)) \\ &= -0.27 \ A, \end{aligned}$$

and

$$i(\infty) = -10 \ A.$$

The time constant is $\tau = L/R = 0.05 \ ms$. Overall, we have

$$\begin{aligned} i(t) &= -10 + (-0.27 + 10)e^{-2 \cdot 10^4 (t - 10^{-4})} \\ &= -10 + 9.73 e^{-2 \cdot 10^4 (t - 10^{-4})} \ A. \end{aligned}$$

4. a. The impedance is given by

$$\begin{aligned} Z(s) &= \frac{1}{sC + \frac{1}{R} + \frac{1}{sL}} \\ &= \frac{s/C}{s^2 + s\frac{1}{RC} + \frac{1}{LC}} \\ &= \frac{s}{s^2 + s + 1}. \end{aligned}$$

b. The two poles are given by $s = -1/2 \pm j\sqrt{3}/2$. There is a zero in $s = 0$.
c. In order to obtain steady-state poles, we need to make the real part zero. This is easily seen to be accomplished when $R \to \infty$. In fact, with this choice the denominator of the impedance becomes $s^2 + 1$ and the poles become $\pm j$.

5. The plot is done as seen during recitation. In particular, we have

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t < 1 \\ 2 - t & \text{if } 1 \leq t < 3 \\ -1 & \text{if } 3 \leq t < 4 \\ -2 + e^{-(t-4)} & \text{if } t \geq 4 \end{cases}.$$

You can check that the function is continuous.