ECE 232 - Circuits and Systems II
Midterm

Please provide clear and complete answers. No points will be granted for plots that do not specify the numerical values on the axes. Also, don't forget to specify the units of measure!

1. (2 points) a. Find the Laplace transform of $5\cos(10t + \pi/2)u(t)$ (Hint: Recall Euler's formula: $\cos x = \frac{e^{jx}+e^{-jx}}{2}$).
b. Find and plot poles and zeros of the Laplace transform found at the previous point.

Sol.: a.

$$\begin{aligned}
& \mathcal{L}(5\cos(10t+\pi/2)u(t)) \\
= & \ \mathcal{L}\left(5\frac{e^{j(10t+\pi/2)} + e^{-j(10t+\pi/2)}}{2}u(t)\right) \\
= & \ \frac{5}{2}e^{j\pi/2}\mathcal{L}\left(e^{j10t}u(t)\right) + \frac{5}{2}e^{-j\pi/2}\mathcal{L}\left(e^{-j10t}u(t)\right) \\
= & \ \frac{5}{2}\frac{j}{s-j10} + \frac{5}{2}\frac{-j}{s+j10} \\
= & \ \frac{5}{2}\frac{j(s+j10) - j(s-j10)}{s^2+100} \\
= & \ 5\frac{-10}{s^2+100} \\
= & \ -\frac{50}{s^2+100}.
\end{aligned}$$

b. There are two poles in ± 10 and no zeros.

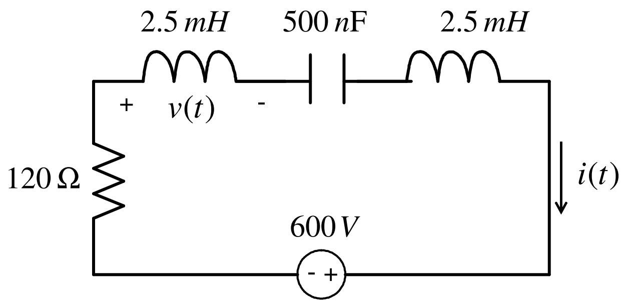
2. (1 point) Consider the parallel of an inductor of inductance 1 H and of a capacitor of capacitance 4 F. Assume that the energy available in the inductor at time $t = 0$ is equal to 2 J, whereas no energy is stored in the capacitor at time $t = 0$. Write the equations of the current flowing in the inductor and capacitor, and of the voltage across inductor and capacitor (choose the direction you would like for the current). Plot these functions.

Sol.: Reasoning as done in class (by using the relationship between voltage and current in the inductor and in the capacitor), the current is a sinusoid with amplitude 2 A that equals 2 A for $t = 0$ and whose frequency is $1/\sqrt{4} = 1/2$ rad/s (i.e., $i(t) = 2\cos(1/2t)u(t)$). The voltage instead is a sinuoid with the same period that starts from zero at $t = 0$ and given as $v(t) = Ldi/dt = -\sin(1/2t)u(t)$.

3. (1 point) Consider a function that is characterized as follows: It is equal to 3 between $t = 0$ and $t = 1$; it equals -1 between $t = 1$ and $t = 3$; and is zero otherwise. Write an equation for this function in terms of the step function $u(t)$.

Sol.: We have

$$\begin{aligned}
& 3u(t) - 3u(t-1) - u(t-1) + u(t-3) \\
= & \ 3u(t) - 4u(t-1) + u(t-3).
\end{aligned}$$



4. (3 points) For the circuit in the figure on the left, we assume that the initial energy in the capacitor is zero, and that $i(0) = -5\ A$.
a. Calculate $i(t)$ for $t \geq 0$.
b. Calculate $v(t)$ for $t \geq 0$.
c. Plot $v(t)$ for $t \geq 0$.

Sol.: a. The series of two inductors has equivalent inductance equal to $L_{eq} = 2.5 + 2.5 = 5$ mH. The system is thus a series RLC circuit. We calculate

$$\begin{aligned}\alpha &= \frac{R}{2L_{eq}} = 12,000 \text{ rad/s} \\ \omega_0 &= \frac{1}{\sqrt{L_{eq}C}} = 20,000 \text{ rad/s.}\end{aligned}$$

The circuit operates in the underdamped regime. The damped frequency is

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 16,000 \text{ rad/s.}$$

We now impose the initial conditions:

$$\begin{aligned}B_1 &= -5 \\ \omega_d B_2 - \alpha B_1 &= \frac{di(0^+)}{dt} = 0,\end{aligned}$$

since $\frac{di(0^+)}{dt} = \frac{1}{L_{eq}}(-600 - 120 \cdot (-5)) = 0$. We then obtain

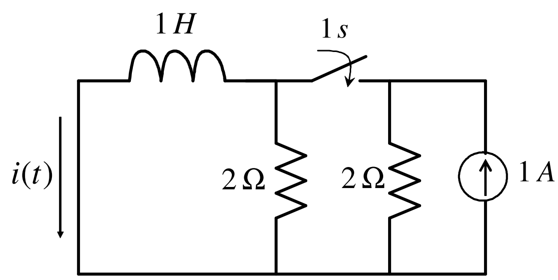
$$B_2 = B_2 \frac{12,000}{16,000} = -\frac{15}{4}$$

$$i(t) = -5e^{-12,000t}\cos(16,000t) - \frac{15}{4}e^{-12,000t}\sin(16,000t),\ t \geq 0.$$

b. We have

$$\begin{aligned}v(t) &= L\frac{di(t)}{dt} \\ &= 2.5 \cdot 10^{-3} \cdot \left(125,000e^{-12,000t}\sin(16,000t)\right) \\ &= 312.5e^{-12,000t}\sin(16,000t),\ t \geq 0.\end{aligned}$$

c. Please see class notes.



5. (3 points) In the circuit on the left, the switch closes at time 1 s and $i(0) = 1$ A.
a. Calculate $i(t)$ for $0 \leq t \leq 1$.
b. Calculate $i(t)$ for $t > 1$.
c. Calculate the fraction of the initial energy in the inductor that is dissipated by the resistor connected to it in the interval $0 \leq t \leq 1$.

a. In the interval $0 \leq t \leq 1$, we have an RL circuit with $\tau = L/R = 1/2$ s. Therefore, we get

$$i(t) = e^{-2t}, \ 0 \leq t \leq 1.$$

b. In the interval $t > 1$, we have an RL circuit with $\tau = L/R_{eq} = 1$ s, where $R_{eq} = 1\ \Omega$. Therefore, we can write

$$\begin{aligned} i(t) &= i(\infty) + (i(1) - i(\infty))e^{-(t-1)} \\ &= 1 + (e^{-2} - 1)e^{-(t-1)}, \text{ for } t > 1. \end{aligned}$$

c. The initial energy in the inductor is 0.5 J. The energy at $t = 1$ is $0.5e^{-4}$. Therefore, a fraction $1 - e^{-4} = 0.98$ (98%) has been dissipated by the resistor.