## ECE 788-Optimization for wireless networks Midterm

Please provide clear and complete answers.
PART I: Questions - Provide a proof or a convincing example to answer the following:
Q.1. (1 point) Is $\left\{(x, t) \in \mathbb{R}^{n} \times \mathbb{R} \mid\|x\|_{2} \geq t\right\}$ a convex set?

Sol.: No. It is enough to sketch the case $n=1$ to recognize this.
Q.2. (1 point) If $C$ and $D$ are convex set, is $C+D=\left\{x=x_{1}+x_{2}\right.$ with $x_{1} \in C$ and $\left.x_{2} \in D\right\}$ convex?
Sol.: To prove that $C+D$ is convex one must verify that for any $x, y \in C+D$ the convex combination $\lambda x+(1-\lambda) y \in C+D$ with $0 \leq \lambda \leq 1$. This is easily proved since we have $x=x_{1}+x_{2}$ and $y=y_{1}+y_{2}$ with $x_{1}, y_{1} \in C$ and $x_{2}, y_{2} \in D$ so that

$$
\begin{aligned}
\lambda x+(1-\lambda) y & =\lambda\left(x_{1}+x_{2}\right)+(1-\lambda)\left(y_{1}+y_{2}\right)= \\
& =\left[\lambda x_{1}+(1-\lambda) y_{1}\right]+\left[\lambda x_{2}+(1-\lambda) y_{2}\right]
\end{aligned}
$$

which belongs to $C+D$ since $\lambda x_{1}+(1-\lambda) y_{1} \in C$ and $\lambda x_{2}+(1-\lambda) y_{2} \in D$ by definition of convexity.
Q.3. (1 point) Can the set $C=\left\{x \in \mathbb{R}^{2} \left\lvert\,\left\|\left[\begin{array}{ll}1 & 2 \\ 3 & 1\end{array}\right]\left[\begin{array}{l}x_{1} \\ x_{2}\end{array}\right]+\left[\begin{array}{l}0 \\ 1\end{array}\right]\right\|_{2} \leq 3 x_{1}+2 x_{2}\right.\right\}$ be written as a Linear Matrix inequality (LMI)? If so, write the corresponding LMI.

Sol.: Yes, defining $A=\left[\begin{array}{ll}1 & 2 \\ 3 & 1\end{array}\right], b=\left[\begin{array}{l}0 \\ 1\end{array}\right]$ and $c=\left[\begin{array}{l}3 \\ 2\end{array}\right]$, the constraint in the definition of $C$ can be written as $\|A x+b\|_{2} \leq c^{T} x$ or $\|A x+b\|_{2}^{2} \leq\left(c^{T} x\right)^{2}$, which is equivalent to

$$
C=\left\{x \in \mathbb{R}^{2} \left\lvert\,\left[\begin{array}{cc}
\left(c^{T} x\right) I_{2} & A x+b  \tag{1}\\
(A x+b)^{T} & c^{T} x
\end{array}\right] \succeq 0\right.\right\}
$$

with $I_{2}=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$. In fact, if $c^{T} x=3 x_{1}+2 x_{2}>0$ (or equivalently $\|A x+b\|_{2}^{2}<\left(c^{T} x\right)^{2}$ ) the two conditions in the original definition of $C$ and in (1) are equivalent due to the Schur complement properties. Moreover, if $c^{T} x=0$ we have (from the definition of $C$ ) that $A x+b=0$, and the resulting all-zero matrix is included in the set (1). We can then conclude that the original definition of $C$ and (1) are equivalent.
Q.4. (1 point) Are these functions convex, concave or neither? (i) $f(x, y)=\exp \left(\frac{x^{2}}{y}\right)$ on $\mathbb{R} \times \mathbb{R}_{++} ;($ii $) f(x)=1 / \log x$ with $x>1 ;($ iii $) f(x)=\left(\prod_{i=1}^{n} x_{i}^{1 / 2}\right)^{1 / n}$ with $x_{i} \geq 0$.
Sol.: $(i) \exp (x)$ is a convex function and non-decreasing, $\frac{x^{2}}{y}$ is convex on $\mathbb{R} \times \mathbb{R}_{++}$(prove, e.g., by using the perspective function property) $\Longrightarrow \exp \left(\frac{x^{2}}{y}\right)$ is convex on $\mathbb{R} \times \mathbb{R}_{++}$.
(ii) $1 / x$ is convex for $x>0$ and its extended-value extension is non-increasing, $\log x$ is concave and non-negative for $x>1 \Longrightarrow 1 / \log x$ is convex for $x>1$.


Figure 1:
(iii) $\left(\prod_{i=1}^{n} x\right)^{1 / n}$ is concave and its extended-value extension is non-decreasing in each argument for $x \geq 0, x_{i}^{1 / 2}$ is concave and non-negative for $x_{i} \geq 0 \Longrightarrow\left(\prod_{i=1}^{n} x_{i}^{1 / 2}\right)^{1 / n}$ is concave for $x_{i} \geq 0$.

## PART II: Problems -

P.1. (2 points) Consider the small network in figure with two sources S1 and S2 that communicate both with destination D. Weights of the edges represent the link capacities. Each source transmits with rate $x_{i} \geq 0(i=1,2)$. We are interested in maximizing a network utility $f_{o}\left(x_{1}, x_{2}\right)$.
(i) Write the optimization problem.

Sol.: Defining

$$
A=\left[\begin{array}{ll}
1 & 0 \\
0 & 1 \\
1 & 1
\end{array}\right], c=\left[\begin{array}{l}
2 \\
2 \\
3
\end{array}\right], x=\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]
$$

we have

$$
\begin{gathered}
\underset{x}{\operatorname{maximize}} f_{o}\left(x_{1}, x_{2}\right) \\
\text { s.t. }\left\{\begin{array}{c}
A x \preceq c \\
x \succeq 0
\end{array}\right.
\end{gathered}
$$

(ii) For each network utility listed in the following, specify if the problem is convex and possibly the type of convex problem. If the problem is convex, evaluate optimal value $p^{*}$ and the set $X_{\text {opt }}$ of optimal solutions. (Hint: sketch feasible set and level curves of the objective functions).
(ii.1) Sum-rate $f_{o}\left(x_{1}, x_{2}\right)=x_{1}+x_{2}$.

Sol.: This is a LP problem. By sketching the feasible region and the level curves of the objective function, it is easy to see that $X_{\text {opt }}=\left\{x \mid x_{1}+x_{2}=3\right.$ and $\left.1 \leq x_{1} \leq 2,1 \leq x_{2} \leq 2\right\}$ and $p^{*}=3$.
(ii.2) Minimum rate $f_{o}\left(x_{1}, x_{2}\right)=\min \left\{x_{1}, x_{2}\right\}$.

Sol.: The problem is convex since the objective is the pointwise minimum of convex (affine) functions and the constraints are all affine. Again, using geometric reasoning, it can be seen that the optimal solution is when $x_{1}=x_{2}$, which leads to $X_{o p t}=\left\{[3 / 23 / 2]^{T}\right\}$ and $p^{*}=3$.
(ii.3) $f_{o}\left(x_{1}, x_{2}\right)=\frac{2}{7} x_{1}^{2}+\frac{2}{5} x_{2}^{2}$.

Sol.: The problem is not convex since the utility is not concave.
(ii.4) $f_{o}\left(x_{1}, x_{2}\right)=-\frac{2}{7} x_{1}^{2}-\frac{2}{5} x_{2}^{2}$.

Sol.: The problem is QP and the optimal value is clearly $p^{*}=0$ with $X_{o p t}=\left\{\left[\begin{array}{ll}0 & 0\end{array}\right]^{T}\right\}$.
P.2. (2 points) (i) Give an explicit solution (optimal value $p^{*}$ and the set $X_{o p t}$ ) for the following problem $(a \neq 0)$

$$
\begin{gathered}
\underset{x}{\operatorname{minimize}} c^{T} x \\
\text { s.t. } a^{T} x \leq b
\end{gathered}
$$

Sol.: By using the necessary and sufficient condition for optimality given convex problems with differentiable objective we have that a point $x^{*}$ is optimal if and only if it satisfies:

$$
\nabla f_{o}\left(x^{*}\right)^{T}\left(y-x^{*}\right) \geq 0 \text { for all } y \in C
$$

where $C$ is the feasible set. In our case, this condition reads:

$$
c^{T}\left(y-x^{*}\right) \geq 0 \text { for } a^{T} y \leq b
$$

By simple geometrical arguments, it can be seen that this is possible if and only if $c$ is parallel to $a$ but with opposite sign, i.e., $c=\lambda a$ with $\lambda \leq 0$. In this case, the optimal set is such that the constraint is active, i.e., $X_{o p t}=\left\{x \mid a^{T} x=b\right\}$ and $p^{*}=\lambda a^{T} x=\lambda b$. If the condition $c=\lambda a$ with $\lambda \leq 0$ is not satisfied, then the problem is unbounded below ( $p^{*}=-\infty$ ) (see problem 4.8 (b)).
(ii) Give numerical values for $p^{*}$ and the set $X_{\text {opt }}$ for the problem at the previous point with $a=\left[\begin{array}{ll}1 & 2\end{array}\right]^{T}, c=[-3-6]^{T}$ and $b=4$. Repeat for $a=\left[\begin{array}{ll}1 & 2\end{array}\right]^{T}, c=[-3-5]^{T}$ and $b=4$.
Sol.: In the first case, the condition derive at the previous point is satisfied with $\lambda=-3$, so that $X_{\text {opt }}=\left\{x \mid x_{1}+2 x_{2}=4\right\}$ and $p^{*}=\lambda b=-12$. In the second case, the condition is not satisfied and therefore we have $p^{*}=-\infty$.
(iii) Give a numerical solution (optimal value $p^{*}$ and the set $X_{o p t}$ ) for the following problem

$$
\begin{gathered}
\underset{x}{\operatorname{minimize}} 3 x_{1}+2 x_{2}+x_{3} \\
\text { s.t. } x \succeq 0, \quad \mathbf{1}^{T} x=1
\end{gathered}
$$

Sol.: We have that $3 x_{1}+2 x_{2}+x_{3} \geq 1$ if $x \succeq 0, \mathbf{1}^{T} x=1$, and this value is attained only for $x=\left[\begin{array}{lll}0 & 0 & 1\end{array}\right]^{T}$. Therefore, we can conclude that $p^{*}=1$ and $X_{o p t}=\left\{\left[\begin{array}{lll}0 & 0 & 1\end{array}\right]^{T}\right\}$.
P.3. (2 points) Consider the uplink channel of a wireless system where the signal-to-noise-plus-interference ratio for the $k$ th user reads

$$
S I N R_{k}=\frac{G_{k} P_{k}}{N+\sum_{j \neq k} G_{j} P_{j}},
$$

where $P_{j}$ are the user powers, $G_{j}$ are the channel power gains and $N$ the noise power. Assume that we have $K=2$ users ( $k=1,2$ ).
(i) Write the problem of maximizing $S I N R_{1}$ (first user) under maximum power constraints (for both users) and minimum SINR constraint for the second user (SIN $R_{2} \geq \gamma$ ). Is it a convex problem? Identify the class to which the problem belongs.

Sol.: The problem is

$$
\begin{aligned}
& \underset{P}{\operatorname{maximize}} \frac{G_{1} P_{1}}{N+G_{2} P_{2}} \\
& \text { s.t. }\left\{\begin{array}{l}
P_{k} \leq P_{k, \text { max }} k=1,2 \\
\frac{G_{2} P_{2}}{N+G_{1} P_{1}} \geq \gamma
\end{array}\right.
\end{aligned}
$$

It is a linear-fractional problem (notice that the second constraint is affine in $P: G_{2} P_{2} \geq$ $\gamma\left(N+G_{1} P_{1}\right)$ ), thus quasi-convex (the objective is quasi-linear). We know that a LFP can be formulated as a LP after appropriate transformations (see p. 151). Also, the problem can be seen as a GP if formulated as

$$
\begin{aligned}
& \underset{P}{\operatorname{minimize}} \frac{N+G_{2} P_{2}}{G_{1} P_{1}}=\frac{N}{G_{1}} P_{1}^{-1}+\frac{G_{2}}{G_{1}} P_{1}^{-1} P_{2} \\
& \text { s.t. }\left\{\begin{array}{l}
\frac{P_{k}}{P_{k, \text { max }} \leq 1 k=1,2} \\
\gamma\left(\frac{N}{G_{2}} P_{2}^{-1}+\frac{G_{1}}{G_{2}} P_{2}^{-1} P_{1}\right) \leq 1 .
\end{array}\right.
\end{aligned}
$$

(ii) Consider now the problem of maximizing the minimum of the two SINRs under maximum power constraints. Answer the same question as above.
Sol.: The problem is

$$
\begin{aligned}
& \underset{P}{\operatorname{maximize} \min }\left\{\frac{G_{1} P_{1}}{N+G_{2} P_{2}}, \frac{G_{2} P_{2}}{N+G_{1} P_{1}}\right\} \\
& \text { s.t. } P_{k} \leq P_{k, \max } k=1,2
\end{aligned}
$$

It is a quasi-convex problem since the constraints are convex (affine) and the objective is the pointwise minimum of quasi-convex functions and thus quasi-convex (alternatively, you can prove quasi-convexity by using the definition, that is, by evaluating the sublevel sets).
(iii) Assume now that we want to maximize the geometric mean of the SINRs ((SIN $R_{1}$. $\left.S I N R_{2}\right)^{1 / 2}$ ) under power constraints. Can the problem be formulated as a convex problem after appropriate transformations? How?

Sol.: Taking the square of the objective function, the problem can be written equivalently as

$$
\begin{aligned}
& \underset{P}{\operatorname{maximize}} \frac{G_{1} P_{1}}{N+G_{2} P_{2}} \cdot \frac{G_{2} P_{2}}{N+G_{1} P_{1}} \\
& \text { s.t. } P_{k} \leq P_{k, \max } k=1,2 .
\end{aligned}
$$

Now, this is equivalent to

$$
\begin{aligned}
& \underset{P}{\operatorname{minimize}} \frac{N+G_{2} P_{2}}{G_{1} P_{1}} \cdot \frac{N+G_{1} P_{1}}{G_{2} P_{2}} \\
& \text { s.t. } P_{k} \leq P_{k, \text { max }} k=1,2
\end{aligned}
$$

which is easily proved to be a GP problem since the objective is a posynomial in $P$.
(iv) Finally, we are interested in minimizing the cost (total power) under quality-of-service constraints on the two users $S I N R_{1} \geq \gamma_{1}$ and $S I N R_{2} \geq \gamma_{2}$. Formulate the problem and define the class it belongs to.

Sol.: The problem reads

$$
\begin{aligned}
& \underset{P}{\operatorname{minimize}} P_{1}+P_{2} \\
& \text { s.t. } \frac{G_{1} P_{1}}{N+G_{2} P_{2}} \geq \gamma_{1}, \frac{G_{2} P_{2}}{N+G_{1} P_{1}} \geq \gamma_{2}
\end{aligned}
$$

and it is easily shown to be LP.

