ECE 788: Network Information Theory
Assignment 1 (due on Sept. 14)

1. Represent all pair of sequences (x^2, y^2) with $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ $(n = 2)$ as points on a plane, as done in class.
1.a. Divide sequences x^2 and y^2 separately into sets of sequences with the same types P_X and P_Y. Calculate the entropy for each type. How many types we have? How many sequences for each type? Check that the theoretical bounds, which depend on the calculated entropies, are satisfied.
1.b. Divide the pair of sequences (x^2, y^2) into sets with the same joint type P_{XY}. Calculate the joint entropy for each joint type. How many joint types we have? How many sequences for each joint type? Check that the theoretical bounds, which depend on the calculated joint entropies, are satisfied.

2. Consider all sequences $x^n \in \mathcal{X}^n$ such that their type $P_X(\cdot) = N(\cdot|x^n)/n$ satisfies the condition

$$H(P_X) \leq R, \tag{1}$$

where R (bits) is a given number. Define such set of sequences as

$$T(R) = \bigcup T_0^n(P_X),$$

where the union is taken over all P_X satisfying (1). Notice that the set $T(R)$ is the set of all sequences whose type has entropy less than R.
2.a. Argue that the number of types P_X that satisfy (1) increases only polynomially with n (Hint: You can immediately prove that this number is less or equal than $(n+1)^{|\mathcal{X}|}$, which is enough).
2.b. Using the result at the previous point, show that the total number of sequences in $T(R)$ satisfies

$$|T(R)| \leq (n+1)^{|\mathcal{X}|} \cdot 2^{nR}. \tag{2}$$

(Hint: Recall that $|T_0^n(P_X)| \leq 2^{nH(P_X)}$)

2.c. How many bits per symbol are need to describe all sequences in $T(R)$? Specifically, calculate an upper bound on such number using (2). Then, take $n \to \infty$, and calculate the number of bits per symbol again. You should be surprised to see that the result is R!

Comment 1: Recall from the lecture that the number of bits per symbol we need to describe *only* the sequences of type $H(P_X)$ is given by $H(P_X)$ for $n \to \infty$. It follows that we need R bits to describe individually any set $T_0^n(P_X)$ with $H(P_X) = R$. The result derived above tells that we need the same amount of bits even if we want to describe *all* sequences in $T(R)$? Is there a contradiction?

Comment 2: The result above can be interpreted by stating that, even if a "receiver" only knows that a certain sequence is in $T(R)$, rather than in a specific $H(P_X)$, we still need the same number of bits (as $n \to \infty$) to describe the sequence to such receiver.