**ECE 788: Network Information Theory**
**Assignment 12 (due on Dec. 7)**

**1.** Consider a wireline network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in which we assign a cost $a(u, v)$ to each edge $(u, v) \in \mathcal{E}$. Sending $x$ packets over edge $(u, v) \in \mathcal{E}$ then costs $a(u, v) \cdot x$ ($x$ can be any real number). Suppose that we want to transmit $R$ packets per edge use from source $s$ to the destinations $d_1, ..., d_{|\mathcal{D}|}$ and that we are interested in minimizing the cost of this operation.
**1.1.** Assuming a single destination $d_1$ (i.e., $|\mathcal{D}| = 1$), justify the reason why the linear program[1] below solves the problem

$$\min_{f(u,v)\geq 0} \sum_{(u,v)\in\mathcal{E}} a(u, v)f(u, v)$$

subject to

$$\sum_{v:\ (u,v)\in\mathcal{E}} f(u, v) - \sum_{u\ (v,u)\in\mathcal{E}} f(v, u) = \begin{cases} R & \text{for } u = s \\ -R & \text{for } u = d_1 \\ 0 & \text{for all other } u \in \mathcal{V} \end{cases} \tag{1}$$

(Hint: If function $f(u, v)$ satisfy the condition above, then $R$ is achievable, how? What is the meaning of variable $f(u, v)$?)
**1.2.** Now consider multiple destinations (i.e., multicasting). First, find $|\mathcal{D}|$ functions $f^{(t)}(u, v) \geq 0$, $t = 1, ..., |\mathcal{D}|$ that satisfy (1) for each destination, i.e.,

$$\sum_{v:\ (u,v)\in\mathcal{E}} f^{(t)}(u, v) - \sum_{u\ (v,u)\in\mathcal{E}} f^{(t)}(v, u) = \begin{cases} R & \text{for } u = s \\ -R & \text{for } u = d_t \\ 0 & \text{for all other } u \in \mathcal{V} \end{cases} \tag{2}$$

Having done this, for each edge, take $\max_t f^{(t)}(u, v)$ to be the capacity of edge $(u, v) \in \mathcal{E}$. Would a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with such edge capacities be able to support a multicast rate of $R$? (Hint: To show this, you have to argue that the min-cut towards each destination is at least $R$, since network coding achieves the min-cut rate for multicasting).
**1.3.** Given the result at the point above, formulate a linear program that solves the minimum cost problem for multicasting (Hint: $z(u, v) = \max_t f^{(t)}(u, v)$ satisfies $z(u, v) \geq f^{(t)}(u, v)$, which is a linear constraint... this requires some thinking).

**2.** Consider a polynomial $P(x_1, x_2, ..., x_N) \in \mathbb{F}[x_1, x_2, ..., x_N]$ for some finite field $\mathbb{F}$, as defined in class. Suppose that $x_i$ are drawn uniformly and independently from $\mathbb{F}$. What can we say about the probability that $P(x_1, x_2, ..., x_N) \neq 0$? Let us prove that $\Pr[P(x_1, x_2, ..., x_N) = 0] \geq \prod_{i=1}^{N}\left(1 - \frac{d_i}{|\mathbb{F}|}\right)$, where $d_i$ is the degree of the polynomial in $x_i$.
**2.1.** Proceed by induction. First prove the result for $N = 1$. (Hint: What is the probability of choosing a root of $P(x_1)$?)
**2.2.** Now, assume that the result is true for $N - 1$ and prove it for $N$ (Hint: A polynomial $P(x_1, x_2, ..., x_N)$ can be written as a polynomial in $x_N$ with coefficients in $P(x_1, x_2, ..., x_{N-1})$).
*Remark*: This result can be used to prove that linear random coding succeeds with high probability if the field size of the packets is large enough (how?).

[1] A linear program is an optimization problem with linear cost function and constraints.