ECE 788: Network Information Theory
Assignment 5 (due on Oct. 19)

1. Consider the following memoryless channel $\mathcal{X} = \mathcal{Y} = \{1, 2, 3, 4\}$,

$$P_{Y|X}(y|x) = \begin{cases} 1/2 & \text{if } y = x \text{ or } y = [x+1]_4 \\ 0 & \text{otherwise} \end{cases},$$

where $[a]_4 = a$ if $a = 1, 2, 3, 4$ and $[5]_4 = 1$ (sketch the channel pmf!).
1.a. Find a simple scheme that achieves $R = 1$ bit/ channel use (Hint: No need for coding! Just transmit an independent bit every symbol with zero error. How?)
1.b. Calculate the capacity $C = \max_{P_X} I(X;Y)$ and verify that the scheme found above is capacity-achieving (Hint: Write $I(X;Y) = H(Y) - H(Y|X)$, where $H(Y|X)$ can be easily calculated and does not depend on P_X).

2. Select one sequence X^n randomly and uniformly within the set $T_\epsilon^n(P_X)$. Then, do the same with a sequence $Y^n \in T_\epsilon^n(P_Y)$. Fix a joint pmf P_{XY} with marginals P_X and P_Y.
2.a. Calculate an upper bound on the probability $\Pr[(X^n, Y^n) = (x^n, y^n)]$ for any given $x^n \in T_\epsilon^n(P_X)$ and $y^n \in T_\epsilon^n(P_Y)$ (Hint: Use the fact that X^n, Y^n are independent and then upper bound $\Pr[X^n = x^n]$ and $\Pr[X^n = x^n]$ using the definition of typicality)
2.b. Following the point above, calculate a lower bound on $\Pr[(X^n, Y^n) = (x^n, y^n)]$ for any given $x^n \in T_\epsilon^n(P_X)$ and $y^n \in T_\epsilon^n(P_Y)$.
2.c. Using the results above, calculate a lower and upper bound on $\Pr[(X^n, Y^n) \in T_\epsilon^n(P_{XY})]$ (Hint: Sum over the probabilities of all $(x^n, y^n) \in T_\epsilon^n(P_{XY})$ and use the known lower and upper bounds on $T_\epsilon^n(P_{XY})$). The result should remind you of the fundamental lemma.

3. Consider a memoryless wireless fading channel

$$Y_i = H_i X_i + Z_i,$$

where H_i is the channel fading gain and Z_i is i.i.d. Gaussian noise with power N, $i = 1, 2, ..., n$. Assume power constraint of P and that H_i has distribution $P_H(0) = 1/2$ and $P_H(1) = 1/2$ and is i.i.d.. In other words, the link is either completely shut down ($H_i = 0$) or $H_i = 1$. Assume that the transmitter does not know the instantaneous value of H_i but the receiver does. This implies that the received signal is equivalently $[Y_i, H_i]$ so that the capacity is

$$C = \max_{f_X:\ E[X^2] \le P} I(X; Y, H),$$

where the maximum is taken with respect to pdf f_X under the constraint $E[X^2] \le P$.
3.a. Show that

$$\begin{aligned} I(X;Y,H) &= h(Y|H) - h(Z) \\ &= h(Y|H) - \frac{1}{2}\log_2(2\pi e N) \end{aligned}$$

(Hint: Use the fact that $h(A, B) = h(A) + h(A|B)$ and similarly for $h(A, B|C)$, and invoke the independence of X and H).

3.b. To calculate the capacity, we thus need to maximize $h(Y|H)$. Show that

$$h(Y|H) = \frac{1}{2}h(Z) + \frac{1}{2}h(X+Z).$$

3.c. Maximize $h(Y|H)$ and prove that

$$\max_{f_X:\, E[X^2]\leq P} h(Y|H) = \frac{1}{4}\log_2(2\pi e N) + \frac{1}{4}\log_2(2\pi e(P+N)).$$

What is the capacity? Interpret the result.
(Additional consideration: If the transmitter knew H_i, it could achieve $\frac{1}{4}\log_2(1+2P/N)$, how? What is the gain with respect to the case where the transmitter does not know H_i?)