ECE 788: Network Information Theory
Assignment 7 (due on Nov. 2)

1. Respond to Marco's concern that a coding scheme based on cosets (or bins or subcodes), rather than simple code multiplexing, may be useful even when both encoder and decoder know the channel state sequence (non-causally for the encoder). To do this, proceed as follows.
1.a. Argue that, if we use code expansion as proposed by Gelfand-Pinsker, the capacity when encoder and decoder know the channel is

$$\max_{P_{U|S},\, X=f(U,S)} I(U;YS) - I(U;S)$$

1.b. Show that the formula found at the previous point is equivalent to

$$\max_{P_{U|S},\, X=f(U,S)} I(U;Y|S).$$

1.c. Write the formula above as a summation over all possible states and argue that it becomes equivalent to

$$\sum_s P_S(s) \max_{P_{X|S}(x|s)} I(X;Y|S=s),$$

i.e., there is no need for the auxiliary codebook.
1.d. Compare the formula above with the rate obtained by using code multiplexing and conclude that the two are the same, and coincide with the capacity with channel state at both encoder and decoder C^{E-D}. This enforces the conclusion that coset-based encoding is not necessary when both encoder and decoder know the state sequence.
(Remark: Start with Shannon's formula for $C^{E-causal}$ and add state information at the receiver. You shoul be able to recover again C^{E-D}, since causal and non-causal information have the same effect when the decoder has state information).

2. Consider a scalar channel $Y = X + Z$ where Z is uniform in the interval $[0, \Delta]$ for some $\Delta > 0$.
2.a. Propose a simple constellation (b_1, b_2) on the real line ($n = 1$) that satisfies the power constraint $E[X^2] \leq \Delta^2/2$ and has zero probability of error on this channel.
2.b. Say that we now have an interference S as $Y = X + S + Z$, where S has some power σ_s^2 and arbitrary distribution. Use the same constellation at the step above and transmit $X = b_i - S$. What would be the probability of error of this scheme? What would be the required power?
2.c. Consider now an alternative scheme based on code expansion. Show that if the transmitter sends $X = (b_i - S) \bmod (2\Delta)$, and the decoder decodes over an expanded alphabet, the probability of error is still zero. Sketch the expanded alphabet.
2.d. Provide an example of the behavior of the above scheme for $S = -3.7\Delta$ (i.e., show that the decoder is able to decode with zero probability of error).
(Remark: Recall the similar scheme used for Wyner-Ziv compression, i.e., for binning in the previous homework)

3. Consider a state-dependent channel $P_{Y|X,S}$ with binary alphabets for all variables at hand. Consider the following model: If $S=0$, the channel is a BSC with bit-flipping probability p, while if $S=1$ the output is stuck at $Y=1$.

3.a. Formalize the model above by defining the function $P_{Y|X,S}(y|x,s)$ for all values of the argument.

3.b. Consider the channel state known non-causally at the transmitter. Justify the following choice of variables in the Gelfand-Pinsker formula, based on the intuition regarding the operational meaning of such variables –

$$\begin{aligned} P_{U|S}(1|1) &= 1,\ X = f(U,1) = 1 \\ P_{U|S}(1|0) &= 1/2,\ X = f(U,0) = U. \end{aligned}$$

3.c. Evaluate the Gelfand-Pinsker rate $I(U;Y) - I(U;S)$ (It is convenient to break the calculation into individual entropies).