ECE 788: Network Information Theory
Assignment 8 (due on Nov. 9)

1. Consider the binary symmetric broadcast channel, defined by the marginals $P_{Y_1|X}$ and $P_{Y_2|X}$ as follows

$$Y_j = X \oplus Z_j,$$

with Z_j such that $P_{Z_j}(1) = p_j$, $j = 1, 2$.

1.a. Show that this channel is (in general stochastically) degraded. To do this, argue that the channel described by

$$\begin{aligned} Y_1 &= X \oplus Z_1, \\ Y_2 &= Y_1 \oplus Z_2'. \end{aligned}$$

has the same marginals as the original channel. Find the distribution of Z_2' that guarantees such equivalence (Hint: Z_2 must have the same distribution as $Z_1 \oplus Z_2'$).

1.b. Sketch the equivalent channel above as a cascade of two binary symmetric channels $X - Y_1 - Y_2$ (Use the usual representation for binary symmetric channels in terms of transitions between inputs and outputs).

1.c. In the region of rates $(R_1, R_0 + R_2)$ sketch the set of rates achievable via TDM (Hint: What is the maximum rate achievable by encoder 1 or encoder 2 alone?)

1.d. We now see whether we can do better by superposition coding. Similarly to the Gaussian case, consider the transmission of $X = U \oplus V$ with U, V independent and $P_U(1) = 1/2$ and $P_V(1) = q$. Suppose that U carries information for decoder 2, who treats V as noise. Sketch the equivalent channel seen by decoder 2 from U to Y_2 in terms of a single binary symmetric channel (Hint: What is the equivalent noise seen at decoder 2? Translate the noise into a standard diagram for a binary symmetric channel).

1.e. Based on the point above, write the achievable rate $R_0 + R_2$ (Hint: Calculate $I(U; Y_2)$).

1.f. Noticing that decoder 1 can cancel U (since it can decode it), find the equivalent (binary symmetric) channel between V, which carries information for decoder 1, and Y_1. Calculate the achievable rate (Hint: You should find that it equals $H(V \oplus Z_1) - H(Z_1) = ...$).
(Additional: Plot the obtained region $(R_1, R_0 + R_2)$ in MATLAB by varying q from 0 to 1/2. Compare with the TDM region)

2. Consider the Gaussian broadcast channel. Start with the general achievable rate region achievable with superposition (see derivation in class)

$$\begin{aligned} R_1 &\leq I(X; Y_1|U) \\ R_0 + R_2 &\leq I(U; Y_2) \\ R_0 + R_1 + R_2 &\leq I(X; Y_1). \end{aligned}$$

Show that, by setting $X = U + V$ with U, V independent and $U \sim \mathcal{N}(0, \alpha P)$ and $V \sim \mathcal{N}(0, (1 - \alpha)P)$ in the expressions above, one obtains the rate region obtained from first principles in class, that is,

$$\begin{aligned} R_1 &\leq \frac{1}{2} \log_2 \left(1 + \frac{\alpha P}{N_1}\right) \\ R_0 + R_2 &\leq \frac{1}{2} \log_2 \left(1 + \frac{\alpha P}{N_2 + (1 - \alpha)P}\right). \end{aligned}$$

Why is the third inequality (on $R_0 + R_1 + R_2$) not necessary?
(Hint: $h(A + B|B) = h(A) = 1/2 \log_2(2\pi e \sigma_A^2)$ if A, B independent and $A \sim \mathcal{N}(0, \sigma_A^2)$.)

3. Consider again the Gaussian broadcast channel. Assume that the transmitter sends $X^n = U^n + V^n$, where U^n is the codeword destined to, say, decoder 2, and V^n (to be designed) carries information for decoder 1. Assume that U^n is generated i.i.d. from $\mathcal{N}(0, \alpha P)$.
3.a. Argue that the channel at hand, from the point of view of V^n can be seen as a point to point Gaussian channel with side information at the encoder about the interference (i.e., the dirty paper coding problem).
3.b. Given the above, what is the maximum rate you expect to be able to obtain from V^n? Compare it with the rate achievable with superposition coding.