ECE 788: Network Information Theory
Assignment 9 (due on Nov. 16)

1. Consider a deterministic broadcast channel $Y_1 = g_1(X)$ and $Y_2 = g_2(X)$ with $f_i(\cdot)$ given functions. The capacity region for this channel was found in class (see (7.20) in the booklet). Notice that capacity region needs to be evaluated by taking the union over all possible P_X.
1.1. Find the distribution $P_{U_1U_2}$ and function $X = f(U_1, U_2)$ that achieve such region for a given P_X in the general Marton's achievable region (see (7.18) in the booklet). In other words, find the assignment $P_{U_1U_2}$ and $X = f(U_1, U_2)$ that reduces (7.18) to (7.20) (Hint: Fix a P_X...)
Let us see how we can achieve the corner rates $R_1 = H(Y_1)$ and $R_2 = H(Y_2|Y_1)$. For this, consider all the typical sequences $T_0^n(P_X)$ for some chosen P_X and large n (How many such sequences do we have?). Notice that a given P_X induces a $P_{Y_1Y_2}$.
1.2. How many distinct pairs of sequences (Y_1^n, Y_2^n) corresponds to the input sequences in $T_0^n(P_X)$? How many distinct sequences Y_1^n? How many distinct sequences sequences Y_2^n? How many distinct sequences sequences Y_2^n that appear with a given $Y_1^n = y_1^n$?
1.3. From the results at the point above, it follows that, to achieve $R_1 = H(Y_1)$, we need to assign a different index w_1 to all pairs with distinct Y_1^n. What is the maximum R_2 then and why? (Hint: It should follow immediately from the previous point).

2. Consider a Gaussian multiple access channel (MAC). Assume that the decoder is only interested in W_1 and not in W_2. We want to find a set of rates (R_1, R_2) for which W_1 can be decoded correctly, while W_2 may or may not be decoded successfully (in other words, the probability of error is $P_e^n = \Pr[\hat{W}_1 \neq W_1]$). We expect this rate region to be larger than the MAC capacity region found in class (where both W_1 and W_2 needed to be decoded). To this end, consider a receiver that runs two decoders: 1) A joint typicality decoder for both (W_1,W_2), as studied in class; 2) A decoder that treats $X_2^n(W_2)$ as noise and decodes only W_1.
2.1. What is the achievable rate region for such receiver (i.e., the rate region for which W_1 gets decoded with vanishingly small probability of error)?
2.2. For what rates R_2 should the receiver run decoder 1 or 2?

3. We have seen that the capacity of a broadcast channel depends only on the marginals $P_{Y_1|X}$ and $P_{Y_2|X}$. As an example, for the Gaussian broadcast channel

$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= X + Z_2, \end{aligned}$$

with $[Z_1, Z_2] \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} N_1 & \rho\sqrt{N_1N_2} \\ \rho\sqrt{N_1N_2} & N_2 \end{bmatrix}\right)$ (and, say, $N_1 \leq N_2$), the capacity is the same irrespective of the correlation coefficient ρ ($-1 \leq \rho \leq 1$). In this example we show that if (Y_1, Y_2) are processed by a unique decoder to recover a common message W_0, then ρ matters!
Consider then a point to point system where the receiver has available (Y_1, Y_2).
3.1. Calculate the capacity of such system by maximizing $I(X; Y_1Y_2) = h(Y_1Y_2) - h(Y_1Y_2|X)$ (Hint: You need the maximum entropy theorem on p. 178. You can simplify the result by doing some algebraic manipulations, but this is not required here).

3.2. Plot (with MATLAB) the capacity above for $-1 \le \rho \le 1$. What happens at $\rho = -1$? What happens at $\rho = 1$?