Math 222 Exam 3, April 13, 2016

Read each problem carefully. Show all your work. Turn off your phones. No notes, books, and calculators.

1. (7 points) Find α so that $y_1 = x^{1/2}$ is a solution to the differential equation
$$x^2y'' + \alpha xy' + y = 0.$$
Find the other linearly independent solution y_2.

2. (8 points) Consider the Legendre equation of order β:
$$(1 - x^2)y'' - 2xy' + \beta(\beta + 1)y = 0.$$
Find all the singular point(s) and determine if they are regular or not.

3. (15 points) Consider a series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the equation
$$y'' + y = 0, \quad -\infty < x < \infty.$$
First find the recurrence relation. Then write out the first two non-zero terms in the two linearly independent series solutions. Determine the radius of convergence for each of the two series by using the ratio test.

4. Find the inverse Laplace transform for (a) and (b): (a) (8 points) $F(s) = \frac{s^2-9}{s^3+6s^2+9s}$. (b) (9 points) $G(s) = \frac{e^{-s}(s-2)}{s^2+2s+2}$. Find the Laplace transform for (c): (c) (8 points) $h(t) = t^2 \sin^2(t)$.

5. Given the function
$$g(t) = \begin{cases} e^{-t} & 0 \le t < 1 \\ e^{-3t} + 1 & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases}.$$
(a) (4 points) Graph the function $g(t)$ for $0 \le t \le 3$.
(b) (4 points) Write $g(t)$ in terms of unit step functions.
(c) (7 points) Find the Laplace transform of $g(t)$.

6. (15 points) Solve the following initial value problem: $y'' + 4y' + 8y = 2u_\pi(t) - 2\delta(t - 2\pi)$, $y(0) = 2$, $y'(0) = 0$.

7. Use the convolution integral for the following problems.
(a) (7 points) Find the inverse Laplace transform of $F(s) = \frac{1}{s^3(s^2+1)}$, and express your answer in a convolution integral (DO NOT integrate it).
(b) (8 points) Find the Laplace transform of $f(t) = \int_0^t (t - \tau)^2 \cos(2\tau)d\tau$.