Practice Exam for M611, Fall 2012

1. Many linear systems $Ax = b$ are directly written in the form $x = b + Mx$ with $A = I - M$. Do a convergence analysis for

$$x^{(k+1)} = b + Mx^{(k)}, \quad k \geq 0$$

2. Consider the iteration

$$x^{(k+1)} = b + \alpha \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x^{(k)}, \quad k = 0, 1, 2, ...$$

where α is a real constant. Find the values of α for which the above iteration converges.

3. Find a and b by minimizing the root-mean-square-error

$$E = \left[\frac{1}{n} \sum_{j=1}^{n} (a + be^{-x_j} - y_j)^2 \right]^{1/2}.$$

Find a linear system satisfied by the optimum choice of a and b.

4. Let $A = \begin{bmatrix} 5 & -2 \\ -1 & 4 \end{bmatrix}$. Apply the power method with $z^{(0)} = [1, 1]^T$. Determine the sequence of $\lambda^{(m)}$ for $m = 5$. Is the method convergent to the dominant eigenvalue and the corresponding eigenvector?

5. Consider a general system of linear ODEs with constant coefficients $\frac{dy}{dt} = f(y) = Ay$, where A is a constant $n \times n$ matrix. If A is a diagonalizable matrix, then its eigenvectors (for eigenvalues $\lambda_1, \ldots, \lambda_n$) form a basis in the n-dimensional space. The ODE system with large differences in eigenvalues is called a stiff ODE system. In this case, semi-implicit method or implicit method is preferred to solve the stiff system of linear ODEs. For example, a semi-implicit method can be developed as

$$y^{(k+1)} = y^{(k)} + \frac{\delta t}{2} (f(t^{(k)}, y^{(k)}) + f(t^{(k+1)}, y^{(k+1)})).$$

Apply this method to the equation $y' = -\lambda y$ with $\lambda > 0$ and examine the stability of the method.

6. Write a pseudo-code to solve the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x), \quad \text{for} \quad -\pi < x < \pi$$

with boundary conditions $u(-\pi) = 0 = u(\pi)$ and initial condition $u(x, t = 0) = g(x)$.