Muller's method:

Given three points : $(x_i, f(x_i))$ $x_i = x_0, x_1, x_2$

construct a quadratic polynomial

$$p(x) = f(x_2) + (x-x_2) f[x_2, x_1] + (x-x_2)(x-x_1) f[x_2, x_1, x_0] \quad — (*)$$

Check: $p(x_i) = f(x_i)$ for $i = 0, 1, 2$

Rewrite (*) as :

$$p(x) = f(x_2) + W(x-x_2) + f[x_2, x_1, x_0](x-x_2)^2$$

$$W \equiv f[x_2, x_1] + (x_2 - x_1) f[x_2, x_1, x_0]$$

find root of $p(x)$ that is closest to x_2 :

$$x - x_2 = \frac{-W \pm \sqrt{W^2 - 4f(x_2) f[x_2, x_1, x_0]}}{2 f[x_2, x_1, x_0]}$$

$\therefore$ next iteration

$$x_3 = x_2 - \frac{2 f(x_2)}{W \pm \sqrt{W^2 - 4 f(x_2) f[x_2, x_1, x_0]}}$$

NOTE: sign is chosen to maximize the magnitude of the denominator.

NOTE: Muller's method has an order of convergence ~ 1.84.

Q: Why Muller's method works for complex roots? How about newton's method?

Recall the error analysis for newton's method, we use mean value theorem to estimate the remainder. For Muller's method (similar to secant method), we use intermediate value theorem for error estimate.

Fixed Point Iteration: assume the root that we look for is a soln. to

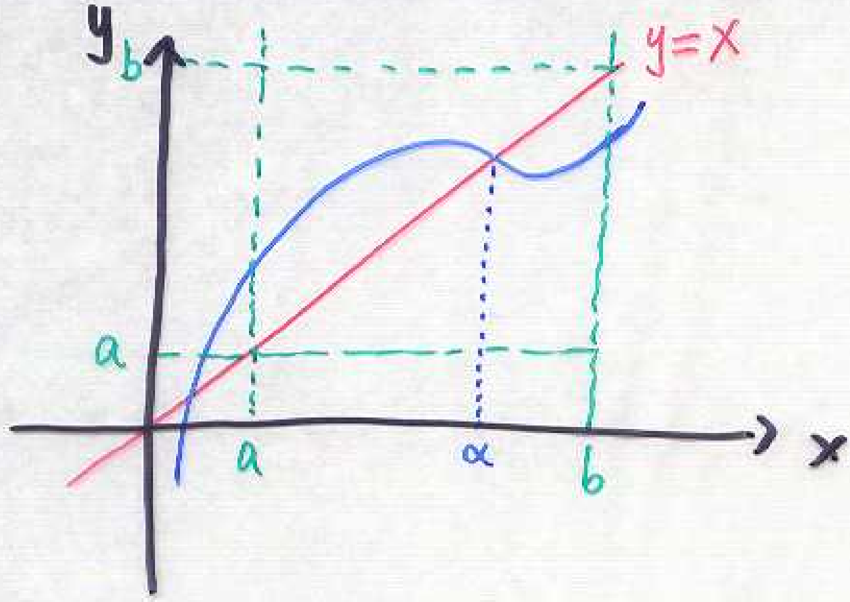
$$x = g(x) \text{ when } x = \alpha.$$

Let $g(x)$ be a continuous function on $[a, b]$ and suppose $a \le g(x) \le b$ for $a \le x \le b$. The equation $x = g(x)$ has at least one soln α in the interval $[a, b]$.

Let $f(x) = x - g(x)$.

$f(a) = a - g(a) \le 0$ → there must be a root by the intermediate value theorem.

$f(b) = b - g(b) \ge 0$


y
b
y=x
a
a
α
b
x

Thm: g & g' are continuous for $a \le x \le b$,

$a \le g \le b$ and $\lambda \equiv \max |g'(x)| < 1$

S1: There is a unique soln α of $x = g(x)$

S2: For any initial guess in $[a, b]$, the iteration will converge to α.

S1: For any two points w & z in $[a, b]$

$g(w) - g(z) = g'(c)(w - z) \quad c \in [a, b]$

$|g(w) - g(z)| = |g'(c)|\,|w - z| \le \lambda\,|w - z|$

if $g(w) = w$
$g(z) = z$

$|g(w) - g(z)| = |w - z| \le \lambda |w - z|, \quad (1-\lambda)|w - z| \le 0$

$w - z = 0$ ※

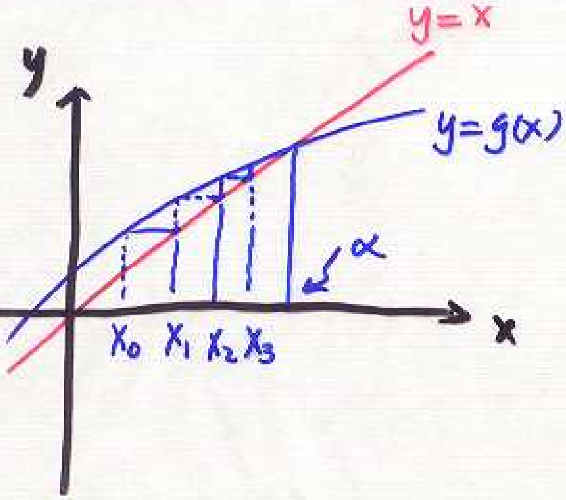
S2: $\alpha - x_{n+1} = g(\alpha) - g(x_n)$

$= g'(c)(\alpha - x_n)$

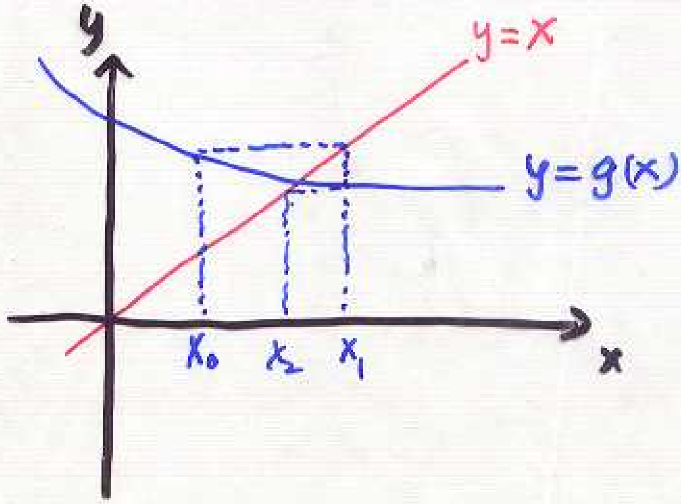
$\therefore |\alpha - x_{n+1}| \le \lambda |\alpha - x_n|$

$\therefore |\alpha - x_{n+1}| \le \lambda^n |\alpha - x_0|$

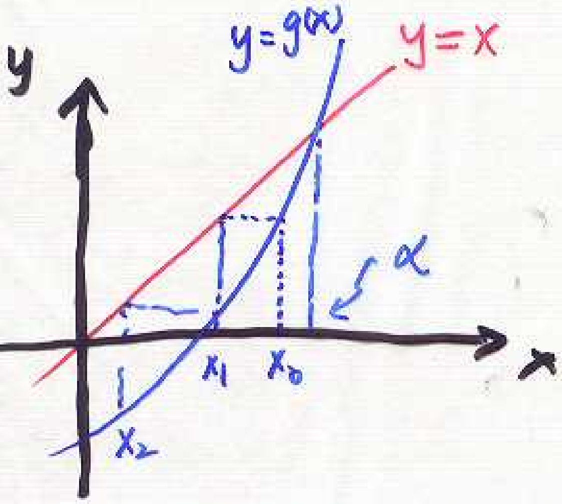
as $\lambda < 1$, $|\alpha - x_{n+1}| \to 0$ as $n \to \infty$ ※



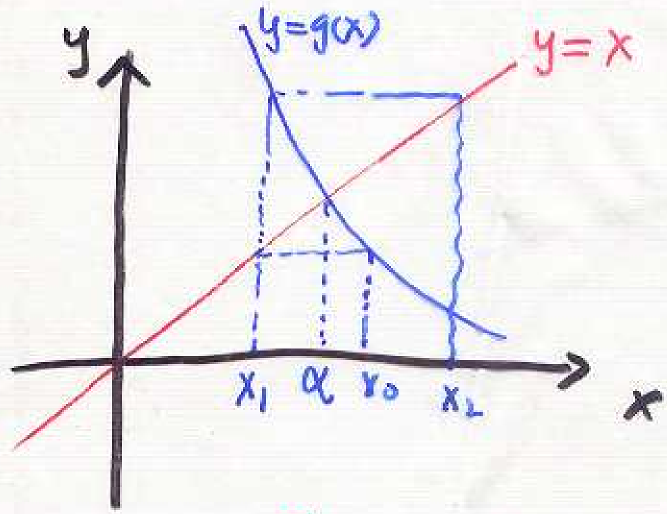
$0 < g'(\alpha) < 1$



$-1 < g'(\alpha) < 0$



$g'(\alpha) > 1$



$g'(\alpha) < -1$

S3: $|\alpha - x_n| \le \dfrac{\lambda^n}{1-\lambda} |x_0 - x_1| \qquad n \ge 0$

S4: $\lim_{n\to\infty} \dfrac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$

S3: for x_n close to α, $\alpha - x_{n+1} \sim g'(\alpha)(\alpha - x_n)$

$$|\alpha - x_0| \le |\alpha - x_1| + |x_1 - x_0|$$
$$\le \lambda|\alpha - x_0| + |x_1 - x_0|$$

$$(1-\lambda)|\alpha - x_0| \le |x_1 - x_0|$$

$$|\alpha - x_0| \le \frac{|x_1 - x_0|}{1-\lambda}$$

since $|\alpha - x_n| \le \lambda^n |\alpha - x_0| \le \dfrac{\lambda^n}{1-\lambda}|x_1 - x_0|$ #

S4: $\lim_{n\to\infty} \dfrac{\alpha - x_{n+1}}{\alpha - x_n} = \lim_{n\to\infty} g'(c_n)$ $\quad c_n$ between α & x_n

because $c_n \to \alpha$ as $n \to \infty$

$\therefore \lim_{n\to\infty} \dfrac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$

order of convergence: $|\alpha - x_{n+1}| \le c|\alpha - x_n|^p \quad n \ge 0$

linear convergence: $p = 1 \quad |c| < 1$

quadratic convergence: $p = 2$

cubic convergence: $p = 3$

Example: Look for root for $x^2 - 5 = 0$

I1: $x_{n+1} = 5 + x_n - x_n^2$

I2: $x_{n+1} = 5/x_n$

I3: $x_{n+1} = 1 + x_n - \frac{1}{5}x_n^2$

I4: $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$

First note that $\sqrt{5}$ is a fixed point:

I1: $x = 5 + x - x^2$

I2: $x = 5/x$

I3: $x = 1 + x - \frac{1}{5}x^2$

I4: $x = \frac{1}{2}(x + 5/x)$

$\alpha = \sqrt{5}$

Detemine $g(x)$ for the above iterations:

I1: $g(x) = 5 + x - x^2$, $g'(x) = 1 - 2x$ $|g'(\alpha)| \geq 1$

I2: $g(x) = 5/x$, $g'(x) = -5/x^2$ $|g'(\alpha)| = 1$

I3: $g(x) = 1 + x - \frac{1}{5}x^2$, $g'(x) = 1 - \frac{2}{5}x$ $|g'(\alpha)| \leq 1$

I4: $g(x) = \frac{1}{2}(x + \frac{5}{x})$, $g'(x) = \frac{1}{2}(1 - \frac{5}{x^2})$ $|g'(\alpha)| = 0$

Define $\lambda_n \equiv \dfrac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}$, $n \geq 2$

as $n \to \infty$, $\lambda_n \to g'(\alpha)$

$$\alpha - x_n = g'(c)(\alpha - x_{n-1}) \sim \lambda_n(\alpha - x_{n-1})$$

$$\therefore \quad \alpha - x_n = \alpha - x_{n-1} + x_{n-1} - x_n$$

$$= \frac{1}{\lambda_n}(\alpha - x_n) + x_{n-1} - x_n$$

$$\therefore \quad \alpha - x_n = \frac{\lambda_n}{1-\lambda_n}(-x_{n-1} + x_n)$$

$$\therefore \quad \alpha = x_n + \frac{\lambda_n}{1-\lambda_n}(x_n - x_{n-1})$$

$$x_{n+1} = x_n + \frac{\lambda_n}{1-\lambda_n}(x_n - x_{n-1})$$

Recall the in the example:

$|g'(\alpha)| = 0$ for I4, which is Newton's method.

$\Rightarrow$ this means the error converges to zero faster than linearly $\rightarrow$ we know $g''(x) = \frac{5}{x^3}$

$$g''(\alpha) \neq 0$$

$\therefore$ the order of convergence is quadratic for Newton's method.

$$g(x_n) = g(\alpha) + g'(\alpha)(x_n - \alpha) + \frac{1}{2!}g''(\xi_n)(x_n-\alpha)^2$$

(say $g'(\alpha) = 0$,)

$$x_{n+1} = \alpha + 0 + \frac{1}{2}g''(\xi_n)(\alpha - x_n)^2$$

$$\frac{\alpha - x_{n+1}}{(\alpha - x_n)^2} = -\frac{1}{2}g''(\xi_n) \Rightarrow \text{quadratic convergence if } g''(\xi_n) \neq 0$$

Note: if $g' = 0 = g''\big|_\alpha$ & $g'''(\alpha) \neq 0$

$\to$ Order 3 convergence.

Example: $g(x) = x - \dfrac{f(x)}{f'(x)}$

$$g'(x) = 1 - \frac{f'^2 - ff''}{f'^2} = \frac{ff''}{f'^2}$$

$g'(\alpha) = 0$ for $f(\alpha) = 0$
$f'(\alpha) \neq 0$

$$g''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)} \neq 0 \Rightarrow \text{2nd order convergence}$$

In general: $g'(\alpha) = 0$, $g''(\alpha) = 0$, $\ldots$ $g^{n}(\alpha) = 0$, $g^{n+1}(\alpha) \neq 0$

$\Rightarrow$ Order of convergence is $n+1$

Multiple Roots:

$$f(x) = (x-\alpha)^m h(x) \quad , \quad h(\alpha) \neq 0$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$f'(x) = m(x-\alpha)^{m-1} h(x) + (x-\alpha)^m h'(x)$$

$$g(x) = x - \frac{(x-\alpha) h}{mh + (x-\alpha)h'}$$

$$g'(\alpha) = 1 - \frac{1}{m} \neq 0 \quad \text{if } m \neq 1$$

$$\therefore \lim_{n\to\infty} \frac{\alpha - x_n}{\alpha - x_{n-1}} = 1 - \frac{1}{m} \quad \Rightarrow \text{linear convergence}$$

$\Rightarrow$ need first to find multiplicity m

$$g'(\alpha) = 1 - \frac{1}{m}$$

$\Rightarrow$ once m is determined, use a modified Newton's method to find multiple roots.

Define a new g:
$$g(x) = x - m \cdot \frac{f(x)}{f'(x)}$$

$$\Rightarrow x_{n+1} = x_n - m \cdot \frac{f(x_n)}{f'(x_n)}$$

Nonlinear System: next week.

Problem 24 (page 121)

Problem 25 (page 122)

for HW2, due Feb 2nd in class.

Example:

Convert $x^2 - 5 = 0$ to

$$x = x + c(x^2 - 5) = g(x)$$

a fixed-point problem

$$g'(x) = 1 + 2cx$$

$$g'(\sqrt{5}) = 1 + 2\sqrt{5}c$$

$$\boxed{|g'(\sqrt{5})| < 1 \quad \lambda < 1} \rightarrow \boxed{-\frac{1}{\sqrt{5}} < c < 0}$$

Example:

iteration $x_{n+1} = 2 - (1+c)x_n + cx_n^3$

converges to $\alpha = 1$ for some c.

What are the values of c for convergence?

For what value of c will the convergence be quadratic?

$g(x) = 2 - (1+c)\,x + cx^3$

$g'(x) = -(1+c) + 3cx^2$

$g'(1) = 3c - (1+c) = 2c - 1$

$|2c-1| < 1, \quad \boxed{0 < c < 1}$

$\begin{cases} g'(1) = 0 & \text{for quadratic convergence} \\ g''(1) \neq 0 \end{cases}$

$g'(1) = 2c - 1 = 0$

$g''(1) = 6c \neq 0$

$\boxed{c = \frac{1}{2} \text{ for quadratic convergence}}$

EXTRA Notes

Newton's method: $f(x) = x^2 - 5$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x_i^2 - 5}{2x_i}$$

$$= \frac{x_i^2 + 5}{2x_i} = \boxed{\frac{1}{2}\left(x_i + \frac{5}{x_i}\right)}$$

$$g(x) = x - \frac{(x-\alpha)h}{mh + (x-\alpha)h'}$$

$$g'(x) = 1 - \frac{h}{mh + (x-\alpha)h'} - \frac{(x-\alpha)h'}{mh + (x-\alpha)h'}$$

$$M = \frac{\max f''(\xi_n)}{2 \min f'(\xi_n)} + \frac{(x-\alpha)h}{(mh + (x-\alpha)h')^2}(mh' + ((x-\alpha)h')')$$

$$|\alpha - x_{n+1}| \le c\,|\alpha - x_n|^p$$

p : order of convergence

p is the power

pth derivative where

$g^{(p)}(\alpha) \neq 0$