Polynomial Interpolation :

Let $x_0, x_1, \dots x_n$ be distinct, and
$y_0, y_1, \dots y_n$ be the corresponding function values

Interpolation:

find $f(x)$ s.t.

$$f(x_i) = y_i \quad , i = 0, 1, \dots , n$$

Polynomial Interpolation:

find a polynomial $p(x)$ s.t.

$$p(x_i) = y_i , \quad i = 0, 1, \dots , n \quad \text{——} \quad (*)$$

$$p(x) = a_0 + a_1 x + \dots + a_n x^n$$

From $(*)$

$$a_0 + a_1 x_0 + a_2 x_0^2 + \dots + a_n x_0^n = y_0$$

$$a_0 + a_1 x_1 + a_2 x_1^2 + \dots + a_n x_1^n = y_1$$

$$\vdots$$

$$a_0 + a_1 x_n + a_2 x_n^2 + \dots + a_n x_n^n = y_n$$

$$\Rightarrow \quad X \cdot a = y$$

$$X = [x_i^j] \quad i, j = 0, 1, \dots, n$$

$$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Further define

$$V_n(x) = \det \begin{bmatrix} 1 & x_0 & x_0^2 & & x_0^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & & x_{n-1}^n \\ 1 & x & x^2 & & x^n \end{bmatrix}$$

$$V_n(x) = \det \begin{bmatrix} 0 & x_0 - x & x_0^2 - x^2 & \cdots & x_0^n - x^n \\ 0 & x_1 - x & x_1^2 - x^2 & \cdots & \\ & x_2 - x & \vdots & & \\ & \vdots & \vdots & & \\ 0 & x_{n-1} - x & x_{n-1}^2 - x^2 & & x_{n-1}^n - x^n \\ 1 & x & x^2 & \cdots & x^n \end{bmatrix}$$

$$= (-1)^{n+2} (x_0 - x)(x_1 - x) \cdots (x_{n-1} - x) \cdot C^{n+1}$$

$$C^{n+1} = \det \begin{bmatrix} 1 & x_0+x & \cdots & x_0^{n-1} + x_0^{n-2}x + \cdots + x^{n-1} \\ 1 & x_1+x & & \vdots \\ \vdots & \vdots & & \\ 1 & x_{n-1}+x & & x_{n-1}^{n-1} + x_{n-1}^{n-2}x + \cdots + x^{n-1} \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & x_0 & & x_0^{n-1} \\ \vdots & x_1 & & x_1^{n-1} \\ & \vdots & & \vdots \\ 1 & x_{n-1} & & x_{n-1}^{n-1} \end{bmatrix} = V_{n-1}(x_{n-1})$$

$\det(\mathbb{X}) \equiv V_n(x_n)$ by definition

$$V_n(x_n) = (x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1}) \cdot V_{n-1}(x_{n-1})$$

$$= (x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})$$
$$(x_{n-1} - x_0)(x_{n-1} - x_1)\cdots(x_{n-1} - x_{n-2})\, V_{n-2}(x_{n-2})$$

$$\vdots$$

$$= \prod_{\substack{i,j \\ 0 \le j < i \le n}} (x_i - x_j)$$

$\Rightarrow$ if x_i are distinct $\Rightarrow$ $\mathbb{X} \cdot a = y$

$\det(\mathbb{X}) \neq 0$

$\Rightarrow$ This proves the uniqueness of polynomial interpolation.

$\Rightarrow$ Explicit form for the polynomial interpolation:

(a) special case when

$$y_i = 1, \quad y_j = 0 \quad \text{for } j \neq i \qquad \text{for some } i, \ 0 \leq i \leq n$$

$$\Rightarrow P(x) = C\,(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)$$

check $p(x_0)=0$

$p(x_1)=0$

$p(x_j,\ j\neq i)=0$

$p(x_i)=y_i=1 \implies C = \left[(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)\right]^{-1}$

$$\therefore \ P(x) = \prod_{j\neq i}\left(\frac{x-x_j}{x_i-x_j}\right) \quad i=0,1,\cdots,n$$

(b) $p(x_i) = y_i$

$$P(x) = \sum_{i=0}^{n} y_i \cdot \prod_{j\neq i}\left(\frac{x-x_j}{x_i-x_j}\right)$$

$\Rightarrow$ This polynomial $p(x)$ passes points

(x_i, y_i) $i=0,1,\cdots n$

and is unique!

To summarize:

$$P(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x) \quad \text{———} \quad (**)$$

$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

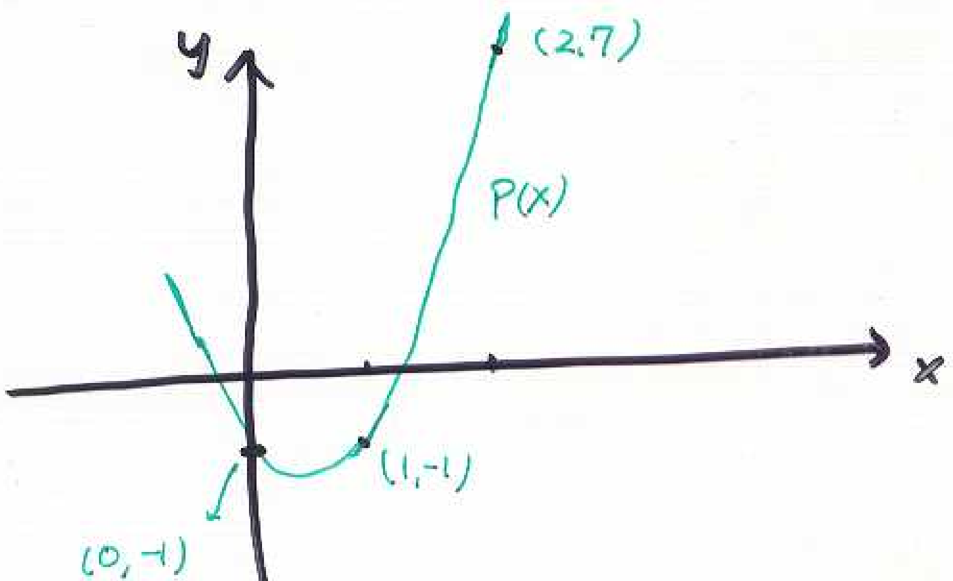
$(**)$ is the Lagrange's formula.

Example:

construct $p(x)$ for the data pts

$(0,-1)$ $(1,-1)$ and $(2,7)$

$$p(x) = (-1)\cdot\frac{(x-1)(x-2)}{2} + (-1)\cdot\frac{(x-0)(x-2)}{-1} + 7\cdot\frac{(x-0)(x-1)}{2}$$



Let $x_0, \cdots, x_n$ be distinct real pts, and consider the following interpolation:

Note: $G(x)$ is $n+1$ times continuosly differentiable,

also $G(x_i) = E(x_i) - \frac{\Phi(x_i)}{\Phi(t)} E(t) = 0$ $i=0,1,\cdots,n$

$$G(t) = E(t) - E(t) = 0$$

$\Rightarrow$ G has $n+2$ distinct zeros in the interval.

G' has $n+1$ distinct zeros

$\vdots$

$G^{(j)}$ has $n+2-j$ zeros

$G^{(n+1)}$ has 1 zero, let it be ξ

$G^{(n+1)}(\xi) = 0$

Recall $G(x) = E(x) - \frac{\Phi(x)}{\Phi(t)} E(t)$

$$G^{(n+1)}(x) = E^{(n+1)}(x) - \frac{\Phi^{(n+1)}(x)}{\Phi(t)} E(t)$$

$$= f^{(n+1)}(x) - \frac{(n+1)!}{\Phi(t)} E(t)$$

$$\therefore \boxed{E(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \Phi(t)}$$

prob 2. prob 12. chap 3

Divided Differences:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(c) \quad c \text{ between } x_0 \,\&\, x_1$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1}{2} f''(c)$$

$$f[x_0, x_1, x_2, x_3] = \frac{1}{x_3 - x_0}\Big(f[x_1, x_2, x_3] - f[x_0, x_1, x_2] \Big)$$

$$\vdots$$

$$f[x_0, x_1, x_2, \cdots x_n] = \frac{1}{n!} f^{(n)}(c)$$

$x_0, x_1, \cdots x_n$ are distinct

f is n times continuously differentiable

c between max and min of x_i

$$\Rightarrow \; f[x_2, x_3, \cdots x_0, \cdots x_n]$$
$$= f[x_0, x_1, x_2, \cdots\cdots x_n]$$

$\Rightarrow$ provided that $f(x)$ is "sufficiently differentiable"

$$f[x_0, x_0, \cdots, x_0] = \frac{1}{n!} f^{(n)}(x_0)$$

Thus: $f[x_0, x_1, x_0] = f[x_0, x_0, x_1]$

$$= \frac{f[x_0,x_1] - f[x_0,x_0]}{x_1 - x_0} = \frac{f[x_0,x_1] - f'(x_0)}{x_1 - x_0}.$$

Newton's divided difference formula for interpolating polynomial:

Let $P_n(x)$ denote the polynomial interpolating $f(x_i)$ at x_i for $i = 0, 1, \cdots, n$

Thus degree of $P_n(x) \le n$ and

$P_n(x_i) = y_i = f(x_i)$, $i = 0, 1, \cdots, n$

$$\Rightarrow P_1(x) = f(x_0) + (x - x_0) f[x_0, x_1]$$

$$P_2(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$

$$\vdots$$

$$P_{k+1}(x) = P_k(x) + (x - x_0) \cdots (x - x_k) f[x_0, x_1, \cdots, x_k, x_{k+1}]$$

$$P_1(x_1) = f(x_0) + (x_1 - x_0)\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= f(x_0) + f(x_1) - f(x_0) = f(x_1)$$

$P_1(x_0) = f(x_0)$, $\deg P_1(x) \le 1$

$\Rightarrow$ $p_1(x)$ is the linear interpolation polynomial to $f(x)$ @ x_0 & x_1.

$P_2(x_0) = f(x_0)$ $\quad\quad \deg P_2(x) \le 2$

$P_2(x_1) = f(x_1)$

$P_2(x_2) = f(x_2)$

$$P_1(x) = f(x_0) + (x - x_0)\, f[x_0, x_1]$$

$$= f(x_0) + \frac{x - x_0}{x_1 - x_0}\left(f(x_1) - f(x_0)\right)$$

$$= f(x_0) + \mu\left(f(x_1) - f(x_0)\right), \quad \mu \equiv \frac{x - x_0}{x_1 - x_0}$$

assuming $\{x_0, x_1, x_2\}$ is evenly spaced

$$x_1 - x_0 = x_2 - x_1 \equiv h, \quad \mu = \frac{x - x_0}{h}$$

$$P_2(x) = P_1(x) + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2]$$

$$P_2(x) = P_1(x) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)} \cdot \Big[f[x_1,x_2] - f[x_0,x_1] \Big]$$

$$= P_1(x) + \frac{h^2\mu(\mu-1)}{2h} \left(\frac{f(x_2)-f(x_1)}{h} - \frac{f(x_1)-f(x_0)}{h} \right)$$

$$= f(x_0) + \mu\,(f(x_1) - f(x_0))$$

$$+ \frac{\mu(\mu-1)}{2} \Big[\underbrace{(f(x_2)-f(x_1))}_{\substack{\text{forward} \\ \text{difference} \\ \equiv \\ \Delta f_1}} - \underbrace{(f(x_1)-f(x_0))}_{\substack{\text{forward} \\ \text{difference} \\ \equiv \\ \Delta f_0}} \Big]$$

$$\Delta^2 f(x_j) \equiv \Delta f_{j+1} - \Delta f_j$$

$$= f_{j+2} - 2f_{j+1} + f_j$$

$$\Delta^k f_j = \Delta^{k-1} f_{j+1} - \Delta^{k-1} f_j \quad \text{for } k \ge 2$$

By definition:

$$\Delta f_j = h \cdot f[x_j, x_{j+1}]$$

further we can show that

$$f[x_j, x_{j+1}, x_{j+2}] = \frac{1}{2h^2}\Delta^2 f_j$$

$$\boxed{f[x_j, x_{j+1}, \dots, x_{j+k}] = \frac{1}{k!\,h^k}\Delta^k f_j}$$

Bonus
HW #3

SCRATCH

$$P_n(x) = \sum_{j=0}^{n} c_j e^{jx} \quad \text{s.t.} \quad P_n(x_i) = y_i$$
$$i = 0, 1, \dots, n$$

Show that there is a unique choice of $c_0, \dots, c_n$

$\Rightarrow$

$$t \equiv e^x$$

$$P_n(t) = \sum_{j=0}^{n} c_j \cdot t^j$$

everything else follows from ordinary polynomial interpolation

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Error Estimate:

Suppose $f \in C^{n+1}([a,b])$ and let $p_n(x)$ be a polynomial of degree $\leq n$. Interpolating f on $[x_0, x_1, \dots, x_n]$ in $[a,b]$. Then for any $x \in [a,b]$ there is a $\xi \in (a,b)$ s.t.

$$f(x) - p_n(x) = \frac{\Phi_n(x) f^{(n+1)}(\xi)}{(n+1)!}$$

$$\Phi_n(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$$

ⓐ if x is a node: $f(x_i) - P_n(x_i) = 0$

ⓑ if x is not a node:

Define $E(x) = f(x) - P_n(x)$

$$G(x) = E(x) - \frac{\Phi(x)}{\Phi(t)} E(t)$$

$$\Phi(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$$

$$\det \begin{vmatrix} x_0 - x & x_0^2 - x^2 & \cdots & x_0^n - x^n \\ x_1 - x & x_1^2 - x^2 & & \vdots \\ \vdots & \vdots & & \vdots \\ x_{n-1} - x & x_{n-1}^2 - x^2 & & x_{n-1}^n - x^n \end{vmatrix}$$

$$= (x_0 - x)(x_1 - x)\cdots(x_{n-1} - x) \cdot$$

$$\det \begin{vmatrix} 1 & x_0 + x & \cdots \\ \vdots & x_1 + x & \\ & \vdots & \\ 1 & x_{n-1} + x & \end{vmatrix}$$

$$\downarrow$$

$$\begin{vmatrix} 1 & x_0 \\ \vdots & x_1 \\ \vdots & \vdots \\ 1 & x_{n-1} \end{vmatrix}$$

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x_i