Example: Let $x_0 < x_1 < x_2$ be given.

Find the polynomial $q(x)$ s.t.

$q(x_0) = y_0$

$q'(x_1) = y_1'$

$q(x_2) = y_2$

What condition, if any, is required in order to have $q(x)$ be uniquely determined from the given interpolation conditions?

$\Rightarrow$ Three conditions $\Rightarrow$ $q(x)$ is quadratic

$$q(x) = Ax^2 + Bx + C$$

$$\begin{aligned} A x_0^2 + Bx_0 + C &= y_0 \\ 2Ax_1 + B &= y_1' \\ Ax_2^2 + Bx_2 + C &= y_2 \end{aligned}$$

$$\cancel{\begin{matrix} x_0^2 & x_0 & 1 \\ 2Ax_1 & 1 & \end{matrix}}$$

$$\begin{bmatrix} x_0^2 & x_0 & 1 \\ 2x_1 & 1 & 0 \\ x_2^2 & x_2 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1' \\ y_2 \end{bmatrix}$$

$$\det \begin{bmatrix} x_0^2 & x_0 & 1 \\ 2x_1 & 1 & 0 \\ x_2^2 & x_2 & 1 \end{bmatrix} \neq 0 \quad \text{for a unique solution}$$

$$\| $$

$$(x_0 - x_2)(x_0 + x_2 - 2x_1)$$

$x_0 \neq x_2$, provided $2x_1 - x_0 - x_2 \neq 0$,

we will have solution

$$\begin{bmatrix} A = \\ B = \\ C = \end{bmatrix} = \begin{bmatrix} x_0^2 & x_0 & 1 \\ 2x_1 & 1 & 0 \\ x_2^2 & x_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1' \\ y_2 \end{bmatrix}$$

Example $P_n(x)$ is the degree n polynomial interpolating to $f(x) = e^x$ on $[a, b] = [0, 1]$ using evenly spaced nodes.
Show that

$$\max_{0 \le x \le 1} \left| e^x - P_n(x) \right| \longrightarrow 0 \quad \text{as } n \to \infty$$

Assume that $x_0, x_1, \cdots x_n$ are evenly spaced in $[0, 1]$

$$\max_{0 \le x \le 1} \left| e^x - P_n(x) \right| = \max_{0 \le x \le 1} \left| \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} e^{c_x} \right|$$

$$\le \frac{1}{(n+1)!} \max_{0 \le x \le 1} \left| \Psi_n(x) \right|$$

$$\max_{0 \le x \le 1} \left| e^{c_x} \right|$$

$$\le \frac{1}{(n+1)!} \; n! \, h^{n+1} \cdot e$$

$$= \frac{e}{n+1} \left(\frac{1}{n} \right)^{n+1} \longrightarrow 0 \quad \text{as } n \to \infty$$

HW4b

Consider computing the interpolating polynomial $P_n(x)$ for the function

$$f(x) = \frac{1}{2 + \sin x}$$ with a uniformly spaced nodes of the

given interval $[a, b]$. Study the ~~interval~~ interpolating polynomial and its error for the interval $[0, 2\pi]$ and $[-\pi, \pi]$ for $n = 10, 20, 30$ & 40.

Spline interpolation :

Consider a grid $a = x_0 < x_1 \cdots < x_n = b$
interpolate $f(x)$ within each interval using a spline function $S(x)$ of order $m \geq 1$ if S satisfies the following two properties

(P1) $S(x)$ is a polynomial of degree $< m$ on each subinterval $[x_{i-1}, x_i]$

(P2) $S^{(r)}(x)$ is continuous on $[a, b]$ for $0 \leq r \leq m-2$

Cubic spline functions (order $m = 4$)

$$S(x) = a_i + b_i x + c_i x^2 + d_i x^3 \qquad x_{i-1} \leq x \leq x_i \qquad i = 1, 2, \cdots, n$$

$$\boxed{S(x_i) = y_i, \quad i = 0, 1, \cdots, n}$$

From (P2) $\boxed{S^{(j)}(x_i+) = S^{(j)}(x_i^-) \qquad j = 0, 1, 2 \quad i = 1, 2, \cdots, n-1}$

All together,

$$n+1 + 3 \cdot (n-1) = 4n-2 \text{ constraints}$$

need two more constraints $\Longrightarrow$ from the end points x_0 & x_n

For example: $S''(x_0) = S''(x_n) = 0$

Within each subinterval, say $[x_{j-1}, x_j]$ we have

$$S''(x_{j-1}) = M_{j-1}$$
$$S''(x_j) = M_j$$

$$S''(x) = \frac{(x_j - x)M_{j-1} + (x - x_{j-1})M_j}{x_j - x_{j-1}} \qquad x_{j-1} \le x \le x_j$$

apply the interpolation conditions $S(x_{j-1}) = y_{j-1}$

$S(x_j) = y_j$

We obtain

$$S(x) = \frac{(x_j - x)^3 M_{j-1} + (x - x_{j-1})^3 M_j}{6(x_j - x_{j-1})} + \frac{(x_j - x)y_{j-1} + (x - x_{j-1})y_j}{x_j - x_{j-1}}$$

$$- \frac{1}{6}(x_j - x_{j-1})\left[(x_j - x)M_{j-1} + (x - x_{j-1})M_j\right]$$

for $x_{j-1} \le x \le x_j$

Continuity of $S'(x)$ gives us the relationship between M_j

$$\frac{x_j - x_{j-1}}{6} M_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} M_j + \frac{x_{j+1} - x_j}{6} M_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

$j = 1, 2, 3, \cdots, n-1$

$M_0 = M_n = 0$

Example: Calculate the cubic spline interpolating the data

$\{ (1,1), (2,\frac{1}{2}), (3,\frac{1}{3}), (4,\frac{1}{4}) \}$.

4 nodes equally spaced : $x_j - x_{j-1} = 1$

$$\therefore \quad \frac{1}{6}M_1 + \frac{2}{3}M_2 + \frac{1}{6}M_3 = \frac{1}{3}$$

$$\frac{1}{6}M_2 + \frac{2}{3}M_3 + \frac{1}{6}M_4 = \frac{1}{12}$$

$$M_1 = M_n = 0$$

$$\therefore \quad M_2 = \frac{1}{2} \qquad M_3 = 0$$

$$\therefore \quad S(x) = \begin{cases} \frac{1}{12}x^3 - \frac{1}{4}x^2 - \frac{1}{3}x + \frac{3}{2} & 1 \le x \le 2 \\ -\frac{1}{12}x^3 + \frac{3}{4}x^2 - \frac{7}{3}x + \frac{17}{6} & 2 \le x \le 3 \\ -\frac{1}{12}x + \frac{7}{12} & 3 \le x \le 4 \end{cases}$$

~~Equation~~

Generally, we have to solve for M in

$$AM = D, \quad \text{where}$$

$$A = \begin{bmatrix} \frac{h_0}{3} & \frac{h_0}{6} & 0 & \cdots & & & 0 \\ \frac{h_0}{6} & \frac{h_0+h_1}{3} & \frac{h_1}{6} & 0 & \cdots & & 0 \\ 0 & \frac{h_1}{6} & \frac{h_1+h_2}{3} & \frac{h_2}{6} & & & \vdots \\ & & & & & & \\ & & & & \frac{h_{n-2}}{6} & \frac{h_{n-2}+h_{n-1}}{3} & \frac{h_{n-1}}{6} \\ & & & & & \frac{h_{n-1}}{6} & \frac{h_{n-1}}{3} \end{bmatrix}$$

$$M^T = [M_0, M_1, \ldots \quad M_n]$$

$$D^T = \left[\frac{y_1 - y_0}{h_0} - y_0', \ \frac{y_2 - y_1}{h_1} - \frac{y_1 - y_0}{h_0}, \ \ldots \ldots \ \frac{y_n - y_{n-1}}{h_{n-1}} - \frac{y_{n-1} - y_{n-2}}{h_{n-2}}, \ y_n' - \frac{y_n - y_{n-1}}{h_{n-1}} \right]$$

$h_0 = x_1 - x_0$

$h_i = x_{i+1} - x_i$

Note: A is symmetric (evenly or unevenly spaced grid)

positive definite

diagonally dominant

Note $AM = D$ is uniquely solvable.

The boundary conditions can be

$$S'(x_0) = f'(x_0) \qquad S'(x_n) = f'(x_n)$$

or

$$S''(x_0) = f''(x_0) \qquad S''(x_n) = f''(x_n)$$

In the previous example:

if we specify $S'(x_0) = f'(x_0)$ $\qquad S'(x_n) = f'(x_n)$

we would have

$$\frac{x_1 - x_0}{3} M_0 + \frac{x_1 - x_0}{6} M_1 = \frac{y_1 - y_0}{x_1 - x_0} - f'(x_0)$$

$$\frac{x_n - x_{n-1}}{6} M_{n-1} + \frac{x_n - x_{n-1}}{3} M_n = -\frac{y_n - y_{n-1}}{x_n - x_{n-1}} + f'(x_n)$$

$\Rightarrow$ obtain this by diff $S(x)$ at subintervals $[x_0, x_1]$, $[x_{n-1}, x_n]$ and then use $S'(x_0) = f'(x_0)$ $\quad S'(x_n) = f'(x_n)$

Example: Calculate the cubic spline interpolating the data

$\left\{ (1,1),\ (2,\tfrac{1}{2}),\ (3,\tfrac{1}{3}),\ (4,\tfrac{1}{4}) \right\}$

with $S'(1) = -1$

$S'(4) = -\frac{1}{16}$

$$\begin{cases} \frac{2M_1}{6} + \frac{M_2}{6} = \frac{3}{6} \\ \frac{M_1}{6} + \frac{4M_2}{6} + \frac{M_3}{6} = \frac{2}{6} \\ \frac{M_2}{6} + \frac{4M_3}{6} + \frac{M_4}{6} = \frac{1}{12} \\ \frac{M_3}{6} + \frac{2M_4}{6} = \frac{1}{48} \end{cases}$$

$$S(x) = \frac{-53x^3 + 332x^2 - 745x + 706}{240}$$

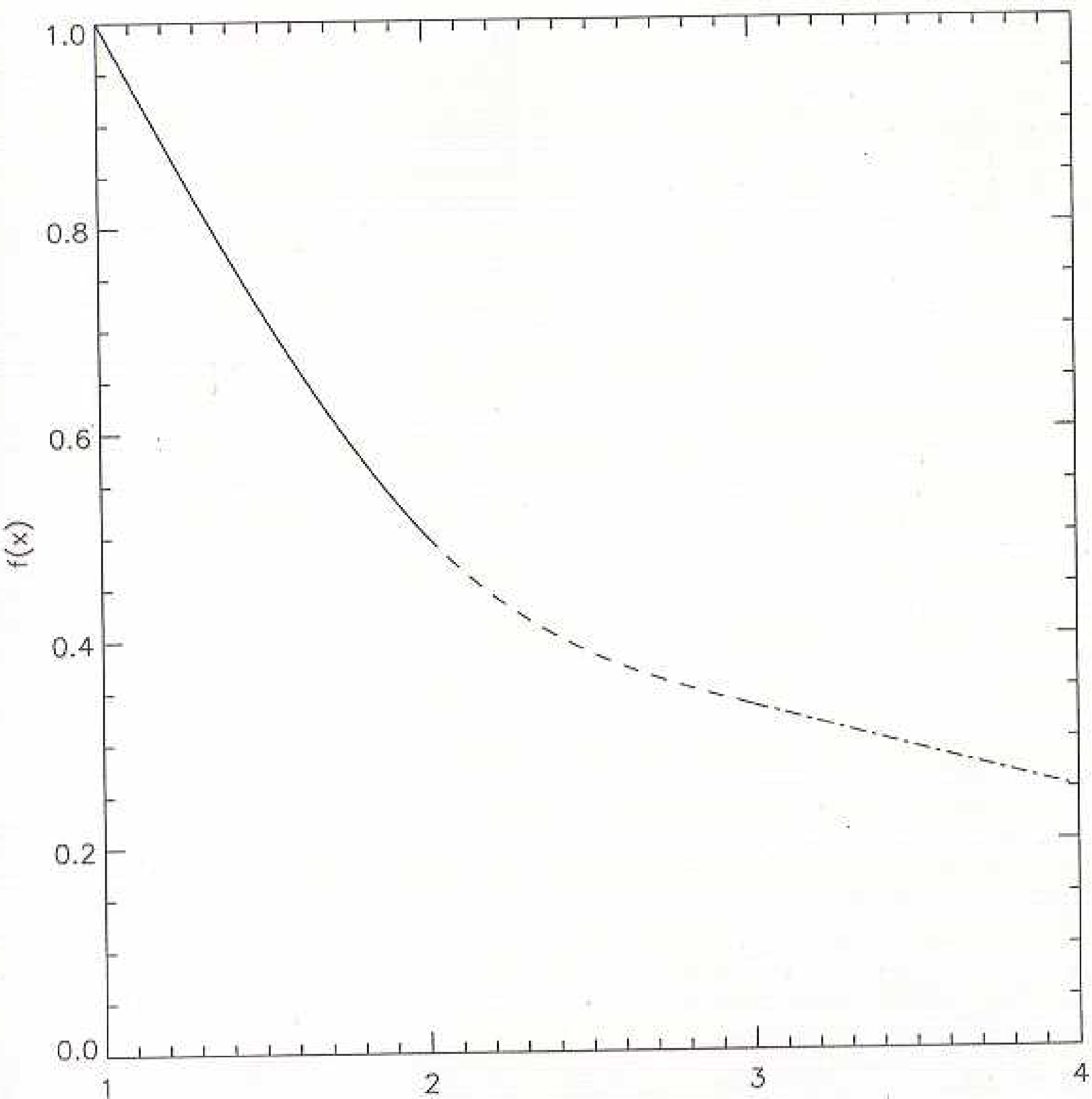
$$\frac{1}{240}\left(-x^3 + 20x^2 - 121x + 290\right)$$

$$\frac{1}{240}\left(-3x^3 + 38x^2 - 175x + 344\right)$$

1st case

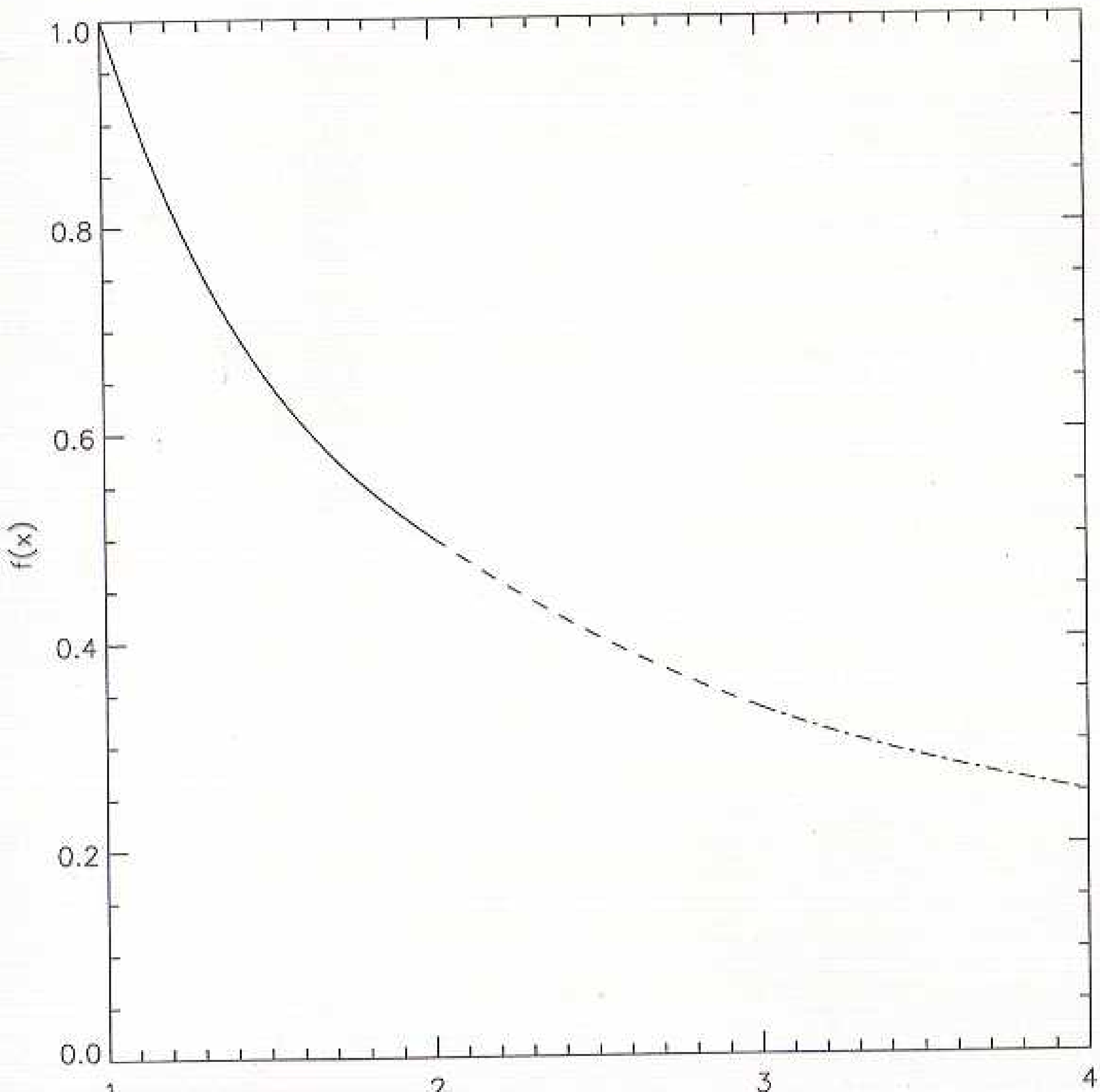
$f''(x_0) = 0$

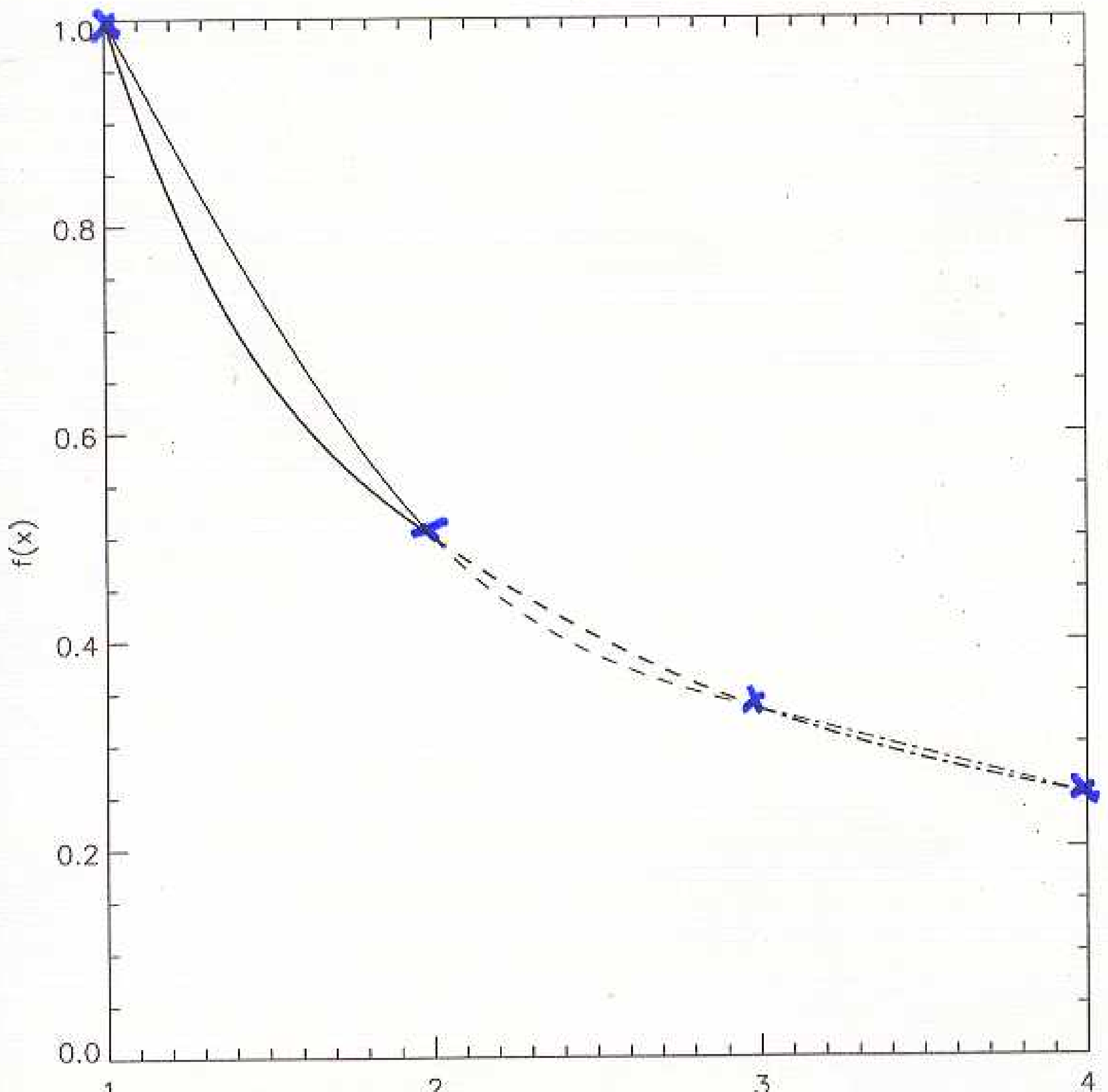
$f''(x_n) = 0$



2nd case:

$f'(x_0)$ & $f'(x_n)$ specified





1.0
0.8
0.6
0.4
0.2
0.0
f(x)
1
2
3
4