Let f have $n+1$ continuous derivatives for x on the interval $[a,b]$. Let $x_i \in [a,b]$ be $n+1$ distinct nodes for $i = 0, 1, \ldots n$.

We know that the interpolating polynomial $P_n(x)$ to $f(x)$ at the nodes satisfies the following equation:

$(P_n(x_i) = f(x_i) \quad \text{for } i = 0, 1 \ldots n)$

$$f(x) - P_n(x) = \frac{\Psi_n(x)}{(n+1)!} f^{(n+1)}(\xi_x), \quad \xi_x \in [a, b)$$

where $\Psi_n(x) = (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)$ is the monic polynomial.

We know that $|\tilde{T}_{n+1}(x)|$ has minimum maximum of absolute value over the domain $[-1, 1]$.

Therefore, one way to minimize the error of interpolating polynomial is to choose the nodes (collocation points) that correspond to roots of $\tilde{T}_{n+1}(x)$

$$|f(x) - P_n(x)| = \frac{|f^{n+1}(\xi_x)|}{(n+1)!} \cdot |\Psi_n(x)|$$

$$E \equiv \max |f(x) - P_n(x)|$$

$$\min_P E = \min \left[\max |f(x) - P_n(x)| \right] = \min \left[\max |\Psi_n(x)| \right] \cdot \frac{|f^{n+1}(\xi_x)|}{(n+1)!}$$

$\Rightarrow$ if we use the Chebyshev nodes, $\min_P E$ will be achieved ~~when~~ $\Rightarrow$ near minimax approximation

Chebyshev collocation points: $x_j = \cos\left(\frac{j\pi}{n+1}\right)$ $j=0,\dots,n+1$

the Chebyshev ~~collocation points~~ nodes are:

$$x_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right) \quad \text{for} \quad T_{n+1}(x)$$

Chebyshev polynomials have the following properties:

$$\varphi_0(x) = \frac{1}{\sqrt{\pi}}, \qquad \varphi_n(x) = \sqrt{\frac{2}{\pi}}\, T_n(x) \quad \text{for } n \geq 1$$

$$(\varphi_n, \varphi_m) = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \qquad \int_{-1}^{1} \frac{\varphi_n \varphi_m}{\sqrt{1-x^2}}\, dx = \delta_{nm}$$

this means that we can write down the polynomial expansions using $T_n(x)$ (or $\varphi_n(x)$) to approximate $f(x)$

$$C_{n+1}(x) = \sum_{j=0}^{n+1} a_j \varphi_j(x), \quad \text{where } a_j \equiv \int_{-1}^{1} \frac{f(x)\varphi_j(x)}{\sqrt{1-x^2}}\, dx$$

$C_{n+1}(x)$ is the "least square approximation" to $f(x)$

What do we mean by "least square" in this context?

Define $G(a_0, \dots, a_n) \equiv \| f - C_n(x) \|_2^2$

$$= \int_{-1}^{1} w(x) \left[f(x) - C_n(x) \right]^2 dx$$

By "least square" we mean to minimize G

$$\frac{\partial G}{\partial a_j} = 0 \qquad j = 0, 1, \cdots, n$$

first $\| f - C_n(x) \|_2^2$

$$= \|f\|_2^2 - 2\sum_{j=0}^{n} a_j (f, \varphi_j) + \sum_i \sum_j a_i a_j (\varphi_i, \varphi_j)$$

$$= \|f\|_2^2 - 2\sum_{j=0}^{n} a_j (f, \varphi_j) + \sum_{j=0}^{n} a_j^2$$

$$= \|f\|_2^2 - \sum_{j=0}^{n} (f, \varphi_j)^2 + \sum_{j=0}^{n} (f, \varphi_j)^2 - 2a_j (f, \varphi_j) + a_j^2$$

$$= \|f\|_2^2 - \sum_{j=0}^{n} (f, \varphi_j)^2 + \sum_{j=0}^{n} \left[(f, \varphi_j) - a_j \right]^2$$

$\Rightarrow$ G is a minimum when $(f, \varphi_j) = a_j$, $j = 0, 1, \cdots, n$

The least square approximation exists, is unique, and has a simple form

$$r_n^*(x) = \sum_{j=0}^{n} (f, \varphi_j) \varphi_j(x)$$

$$\Rightarrow \| f - r_n^* \|_2^2 = \|f\|_2^2 - \|r_n^*\|_2^2$$

$$\|f\|_2^2 = \|f - r_n^*\|_2^2 + \|r_n^*\|_2^2$$

We can further show that

$$\lim_{n\to\infty} \| f - r_n^* \|_2 = 0$$

Compare near-minimax approximation with linear least-square approximation of e^x on $[-1, 1]$

① linear near-minimax approx.

$n=1$, nodes are zeros of $T_2(x)$: $x_1 = \frac{1}{\sqrt{2}}$, $x_2 = -\frac{1}{\sqrt{2}}$

$$P_1(x) = f\left(-\frac{1}{\sqrt{2}}\right) + \frac{x-\left(-\frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}}-\left(-\frac{1}{\sqrt{2}}\right)} \left[f\left(\frac{1}{\sqrt{2}}\right) - f\left(-\frac{1}{\sqrt{2}}\right) \right]$$

$$= e^{-1/\sqrt{2}} + \frac{x + \frac{1}{\sqrt{2}}}{\sqrt{2}} \left(e^{\frac{1}{\sqrt{2}}} - e^{-\frac{1}{\sqrt{2}}} \right)$$

$$= 1.0854\, x + 1.2606$$

$$\max_{-1 \le x \le 1} |e^x - P_1(x)| = 0.372$$

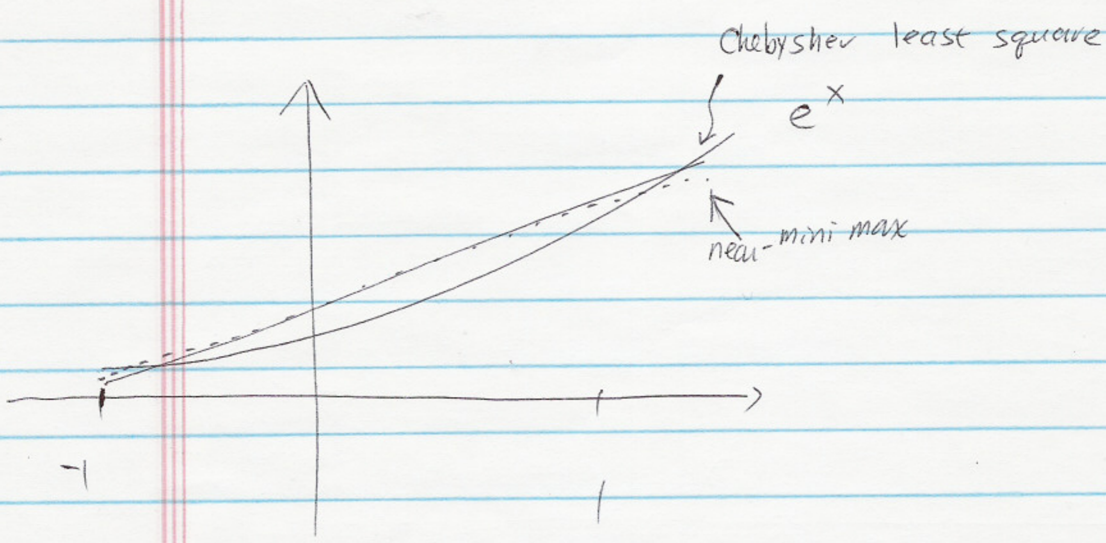
② least-square approx. of e^x using chebyshev polynomials

$$C(x) = a_0 T_0(x) + a_1 T_1(x) + \cdots + a_n T_n(x)$$

$$a_0 = \int_{-1}^{1} \frac{e^x T_0(x)}{\sqrt{1-x^2}} dx \qquad a_1 = \int_{-1}^{1} \frac{e^x T_1(x)}{\sqrt{1-x^2}} dx$$

$C_1(x) = 1.130x + 1.266$

for linear Chebyshev polynomial expansion



$$E_p = \max_{-1 \le x \le 1} |e^x - C_1(x)| = 0.322282$$

③ cubic near – minimax approximation

i	x_i	$D_i f$
0	0.923880	2.616767
1	0.382683	1.146304
2	−0.382683	0.242548
3	−0.923880	0.034780

$$P_3(x) = 2.616767 + (x-x_0)\cdot 1.146304 + (x-x_0)(x-x_1)\cdot 0.242548 + (x-x_0)(x-x_1)(x-x_2)\cdot 0.034780$$

$$\boxed{\max |P_3(x) - e^x| = 0.0006}$$

④ cubic Chebyshev polynomial expansion

$$\cancel{C(x) = 2.53204\,\varphi_0 + 1.12977\,\varphi_1}$$

$$\cancel{x\,1.26607\,\varphi_0 + 1.13031\,\varphi_1 + 0.271450\,\varphi_2 + 0.0437936\,\varphi_3}$$

$$C_3(x) = .994571 + .997308x + .542991x^2 + .177347x^3$$

HW5 is to compare the fifth order near-minimax with fifth order chebyshev polynomial expansion

Practical Issue

Want to do least square approximation on interval $[a, b]$ using Chebyshev polynomials, defined over domain $[-1, 1]$

$\Rightarrow$ change of variable

$$x = \frac{(b+a) + (b-a)t}{2} \qquad \begin{array}{l} x \in [a, b] \\ t \in [-1, 1] \end{array}$$

then do least square on $f\left(\frac{b+a+(b-a)t}{2}\right)$

Generalized least squares

Goal is to minimize $\displaystyle\int_a^b \underset{\substack{\uparrow \\ \text{weight function}}}{w(x)} \left[f(x) - r(x) \right]^2 dx$

properties of weight fcns:

① $w(x) \geq 0 \qquad x \in [a, b]$

② $\displaystyle\int_a^b |x|^n w(x)\, dx$ exists and is finite

③ $\displaystyle\int_a^b w(x) g(x)\, dx = 0$ iff $g(x) = 0$

$w(x) = 1$ $\qquad x \in [a, b]$?

$w(x) = \frac{1}{\sqrt{1-x^2}}$ $\qquad x \in [-1, 1]$

$w(x) = e^{-x}$ $\qquad \rightarrow \; x \in [0, \infty]$

Chebyshev polynomials $\rightarrow$ $w(x) = \frac{1}{\sqrt{1-x^2}}$

Legendre Polynomials $\rightarrow$ $w(x) = \cancel{e^{x}} \; 1$

Laguerre Polynomials $\rightarrow$ $w(x) = e^{-x}$

- Chebyshev polynomial: $\qquad T_n(x) = \cos(n \cos^{-1} x)$

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$$

$$T_0(x) = 1 \qquad T_1(x) = x$$

- Legendre polynomial

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \left[(1-x^2)^n \right] \qquad n \geq 1$$

$$P_0(x) = 1$$

- Laguerre Polynomial

$$L_n(x) = \frac{1}{n!\,e^{-x}} \frac{d^n}{dx^n}\left\{x^n e^{-x}\right\} \qquad n \geq 0$$

$\Rightarrow$ All the above are orthogonal polynomials, meaning each member is orthogonal to every other member of the family.

Thm: Let $\{\varphi_n(x) \mid n \geq 0\}$ be an orthogonal family of polynomials on $[a,b]$ with weight function $w(x)$. With such a family we can always assume that degree $\varphi_n = n$, $n \geq 0$. If $f(x)$ is a polynomial of degree m, then

$$f(x) = \sum_{n=0}^{m} \frac{(f, \varphi_n)}{(\varphi_n, \varphi_n)} \varphi_n(x) \qquad \text{is exact.}$$

$$1 = \frac{1}{c}\varphi_0(x) \qquad \text{if} \quad \varphi_0(x) = c$$

$$\varphi_1(x) = c_{1,1}\, x + c_{1,0}\,\varphi_0(x)$$ because $\varphi_1(x)$ is of degree 1 polynomial

$$\therefore \quad x = \frac{1}{c_{1,1}}\left[\varphi_1(x) - c_{1,0}\,\varphi_0(x)\right]$$

Similarly

$$\varphi_r(x) = c_{r,r}\, x^r + c_{r,r-1}\,\varphi_{r-1}(x) + \dots + c_{r,0}\,\varphi_0(x)$$

$$X^r = \frac{1}{C_{r,r}} \left[\varphi_r(x) - C_{r,r-1}\varphi_{r-1}(x) - C_{r,r-2}\varphi_{r-2}(x) - \cdots - C_{r,0}\varphi_0(x) \right]$$

$\Rightarrow$ every X^r can be expressed as sum of φ_i no greater than $i=r$

$$\therefore \quad f(x) = b_m \varphi_m(x) + \cdots + b_0 \varphi_0(x)$$

$$(f, \varphi_i) = \sum_{j=0}^{m} b_j (\varphi_j(x), \varphi_i)$$

$$= \sum_{j=0}^{m} b_j \cdot \delta_{ij} * (\varphi_i, \varphi_i)$$

$$= b_i (\varphi_i, \varphi_i)$$

$$\therefore \quad b_i = \frac{(f, \varphi_i)}{(\varphi_i, \varphi_i)}$$