M614 02/23/05

Consider interpolating a function within the interval $[-1, 1]$. Let N be an integer and $x_0, x_1, \ldots, x_N$ be a set of distinct points in $[-1, 1]$.

Say $1 \geq x_0 > x_1 > x_2 > \cdots > x_{N-1} > x_N \geq -1$

Equispaced points : $x_j = 1 - \frac{2j}{N}$ $\quad 0 \leq j \leq N$

Chebyshev zero points : $x_j = \cos \frac{(j - 1/2)\pi}{N}$ $\quad 1 \leq j \leq N$

Chebyshev collocation pts : (extreme) $x_j = \cos \frac{j\pi}{N}$ $\quad 0 \leq j \leq N$

Legendre zero points : x_j = jth zero of P_N $(1 \leq j \leq N)$

Legendre extreme points : x_j = jth extreme of P_N $(0 \leq j \leq N)$

density of nodes :

$N \to \infty$ $\quad \mu(x) = \frac{N}{2}$ for equally spaced points

For Chebyshev zero points

$$\frac{\partial x_j}{\partial j} = - \sin(j-\tfrac{1}{2})\pi/N \cdot \left(\frac{\pi}{N}\right)$$

density $\mu(x_j) = -\dfrac{1}{\frac{\partial x_j}{\partial j}} = \dfrac{N/\pi}{\sin(j-\frac{1}{2})\pi/N} = \dfrac{N/\pi}{\sqrt{1-x_j^2}}$

$$\therefore \boxed{\mu(x) = \frac{N}{\pi\sqrt{1-x^2}}}$$

Why is it a good idea to use Chebyshev zeros or extreme pts for grids to interpolate / approximate a function ?

Why not use equally spaced grids ?

If we use (polynomial) trigonometric interpolation in equispaced points, say for an analytic ~~periodic~~ function f in $[-1, 1]$

Gibbs phenomenon : $\|f - P_N\| = O(1)$ as $N \to \infty$
(Runge) even if f is analytic
$= O(2^N)$ for polynomial interpolation

$\Rightarrow$ doubling the domain and reflecting (periodic boundary conditions) can reduce Gibbs phenomenon.

Using Chebyshev points or Legendre points

$\|f - P_N\| \sim O(\text{constant}^{-N})$ if f is analytic

Definition of Lebesgue constant

$$\Lambda_N = \|I_N\|_\infty$$

I_N is the interpolation operator $I_N : f \longmapsto p_N$

A small Lebesgue constant means that the interpolation is not much worse than best (minimax) approx.

$$\| f - p_N \| \leq (\Lambda_N + 1) \| f - p_N^* \|$$

p_N^* is the best (minimax) approx.

Theorem :

For Equispaced points : $\Lambda_N \sim 2^N / eN \log N$

Chebyshev pts : $\Lambda_N \sim \log N$

Legendre pts : $\Lambda_N \sim \sqrt{N}$

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say $f(x;N) = \cos(\alpha N x)$ is an approximation as $N \to \infty$.

$f_N(x)$ changes but the number of points per wavelength remains constant.

Will $\| f_N - p_N \| \to 0$ as $N \to \infty$

For equispaced points : convergence is achieved if

there are at least 6 pts / wave length

For chebyshev points : convergence if π points / wave length
(on average)

on average because the grid is non-uniform.

$\Rightarrow$ $\frac{\pi}{2}$ times less dense in the middle than the equispaced pts
with the same number of N pts.

$\Rightarrow$ in the center, only $\frac{2 \text{ pts}}{\text{wave length}}$ are needed

$\Rightarrow$ familiar Nyquist limit

Example: Show that $X^n = \sum_{j=0}^{n} C_{j,n} P_j(x)$

for appropriate coefficients $\{C_{j,n}\}$

the Legendre polynomial

$P_j(x)$ is an orthogonal polynomial family

because $(P_i, P_j) = \delta_{ij}$

$$\Rightarrow \begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \\ P_n(x) \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}$$

$$= A \cdot \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}$$

$\det(A) \neq 0$

$$\therefore \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix} = A^{-1} \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_n \end{bmatrix} \qquad \therefore C_{ij} = (A^{-1})_{ij}$$

$\Rightarrow P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = -\frac{1}{2} + \frac{3}{2}x^2$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

$$\therefore \boxed{\begin{aligned} 1 &= P_0 \\ X &= P_1 \\ X^2 &= \tfrac{1}{3}P_0 + \tfrac{2}{3}P_2 \end{aligned}}$$

How about $n=3$?

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & 3/2 & 0 \\ 0 & -3/2 & 0 & -5/2 \end{bmatrix} \begin{bmatrix} 1 \\ X \\ X^2 \\ X^3 \end{bmatrix}$$

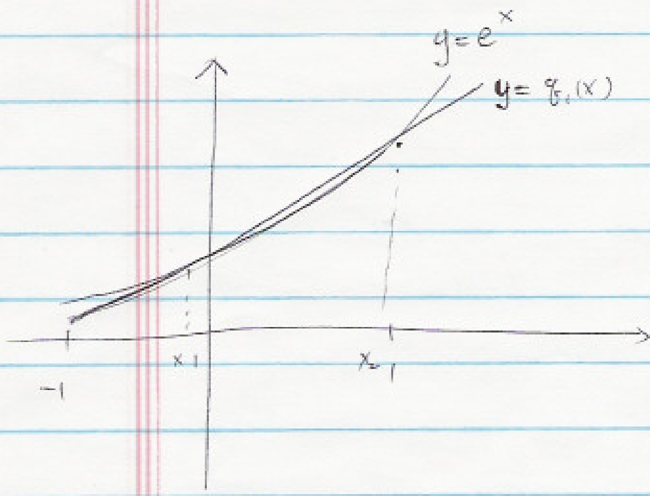
$$\therefore \begin{bmatrix} 1 \\ X \\ X^2 \\ X^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 3/5 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$\therefore \boxed{\begin{aligned} 1 &= P_0 \\ X &= P_1 \\ X^2 &= \tfrac{1}{3}P_0 + \tfrac{2}{3}P_2 \\ X^3 &= \tfrac{3}{5}P_1 + \tfrac{2}{5}P_3 \end{aligned}}$$

Minimax Approximation

$\Rightarrow$ minimax polynomial approx. to e^x on $[-1, 1]$

Let $q_1^*(x) = a_0 + a_1 x$ be the linear minimax approx.



define the error $\varepsilon(x)$ as

$$\varepsilon(x) = e^x - (a_0 + a_1 x)$$

$$\rho_1 \equiv \max_{-1\le x\le 1} (\varepsilon(x))$$

shift $q_1^*(x)$ s.t.

$$\varepsilon(-1) = \varepsilon(1) = -\varepsilon(x_3) = \rho_1$$

with $x_1 < x_3 < x_2$

$\therefore$

$$e^{-1} - (a_0 - a_1) = \rho_1$$
$$e^{x_3} - (a_0 + a_1 x_3) = -\rho_1$$
$$e^{1} - (a_0 + a_1) = \rho_1$$
$$\varepsilon'(x_3) = 0, \qquad e^{x_3} - a_1 = 0$$

four unknowns
four equations

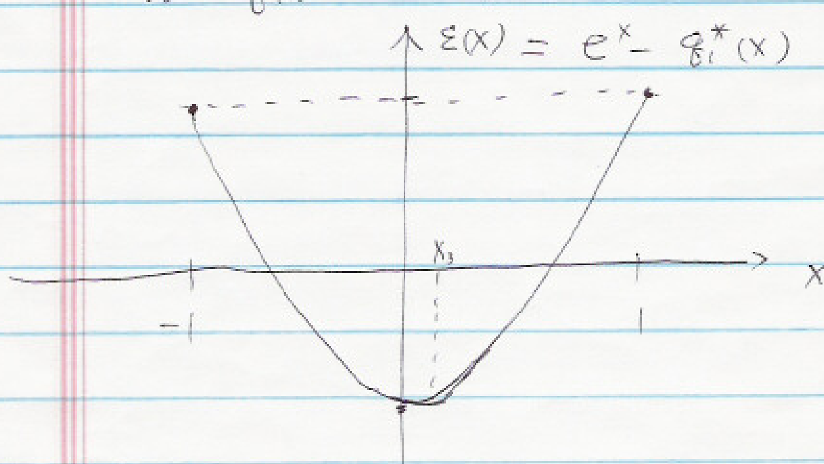
$$\boxed{a_1 = \frac{e^1 - e^{-1}}{2}}, \quad \boxed{x_3 = \ln\left(\frac{e - e^{-1}}{2}\right)}$$

$$e^{x_3} - e - a_1 x_3 + a_1 = -2\rho_1$$
$$2a_1 - e - a_1 x_3 = -2\rho_1, \quad \boxed{\rho_1 = \frac{1}{2}e^{-1} + \frac{x_3}{4}(e - e^{-1})}$$

$\sim .2788$

$$a_0 = \rho_1 + (1 - x_3)a_1$$

$$\therefore \quad q_1^*(x) = 1.2643 + 1.1752x$$



- error is evenly distributed over the interval
- error is oscillating between $+\rho_1$ & $-\rho_1$

Thm : Let $f \in C[a,b]$ and $n \geq 0$.
Suppose we have a polynomial $Q(x)$ of degree $\leq n$ which satisfies

$$f(x_j) - Q(x_j) = (-1)^j e_j \qquad j = 0, 1, \cdots, n+1$$

with all e_j non zero and of the same sign and with
$a \leq x_0 \leq x_1 \quad \cdots \quad < x_{n+1} \leq b$

Then $\displaystyle \min_{0 \leq j \leq n+1} |e_j| \leq \rho_n(f) \equiv \underbrace{\| f - q_n^* \|_\infty}_{\text{by definition}} \leq \| f - Q \|_\infty$

Assume $\displaystyle \rho_n(f) < \min_{0 \leq j \leq n+1} |e_j|$

By definition of $\rho_n(f)$, there exists a polynomial $P(x)$ of degree $\leq n$ s.t.

$$\rho_n(f) \leq \| f - P \|_\infty < \min |e_j|$$

$\boxed{\text{all } e_j > 0}$

$\Rightarrow R(x) \equiv Q(x) - P(x)$ a polynomial of degree $\leq n$.

$$R(x_0) = Q(x_0) - P(x_0) = \left[f(x_0) - P(x_0) \right] - \left[f(x_0) - Q(x_0) \right]$$
$$= \left[f(x_0) - P(x_0) \right] - e_0 < 0$$

Next $R(x_1) = Q(x_1) - P(x_1) = \left[f(x_1) - P(x_1) \right] - (-e_1) > 0$

Therefore the sign of $R(x_j)$ is $(-1)^{j+1}$ for $j = 0, 1, \cdots n+1$

$\Rightarrow R(x)$ has $n+1$ zeros $\Rightarrow$ since $\deg(R) \leq n$, $R \equiv 0$.

$P \equiv Q$

This contradicts $\rho_n(f) \le \|f - P\|_\infty < \mathrm{Min}\,|e_j|$

$\Rightarrow$ because $\cancel{\rho_n(f)})$

$$f(x_j) - Q(x_j) = (-1)^j e_j$$

$\Rightarrow$ therefore $\rho_n(f) \ge \min\limits_{0 \le j \le n+1} |e_j|$

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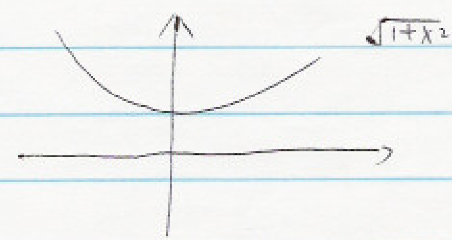
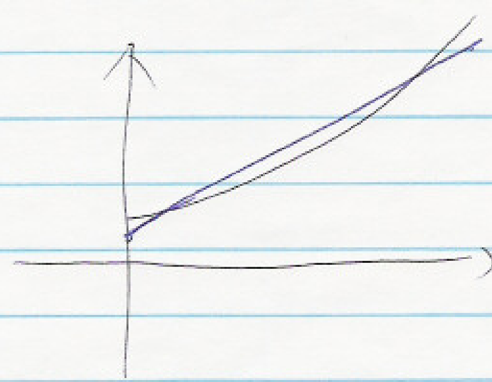
(a) show that the linear minimax approx. to $\sqrt{1+x^2}$ on $[0,1]$ is $q_1^*(x) = .955 + .414x$



write $q_1^*(x) = a_0 + a_1 x$



$$\sqrt{1+1} - (a_0 + a_1) = \rho_1$$

$$\sqrt{1+0} - (a_0 + 0) = \rho_1$$

$$\sqrt{1+x_3^2} - (a_0 + a_1 x_3) = -\rho_1$$

$$\frac{1 \cdot 2x_3}{2\sqrt{1+x_3^2}} - a_1 = 0$$

$\boxed{a_1 = \sqrt{2} - 1 = .414}$, $x_3 = \left(\frac{1}{2+2\sqrt{2}}\right)^{1/2} = 0.45509$

$$a_0 = \frac{1}{2}\left(1 + \sqrt{1+x_3^2} - a_1 x_3\right)$$

$$\boxed{a_0 = .95514}$$

$$\rho_1 = 1 - a_0 = .04486$$

Prob11 (b) $\sqrt{y^2+z^2} = z\cdot\sqrt{1+(y/z)^2} \qquad 0 \le y \le z$

$$\sqrt{1+(y/z)^2} \sim q_1^*(y/z) = .955 + .414\left(\frac{y}{z}\right)$$

$$z\cdot\sqrt{1+(y/z)^2} \sim z\cdot q_1^*(y/z) = .955z + .414y$$

the error $\quad z\cdot\rho_1 = z\cdot .04486 \quad$ is a function of z.

It is easy to remember how Chebyshev points are defined: they are the projections onto the interval $[-1,1]$ of equally-spaced points (roots of unity) along the unit circle $|z|=1$ in the complex plane:

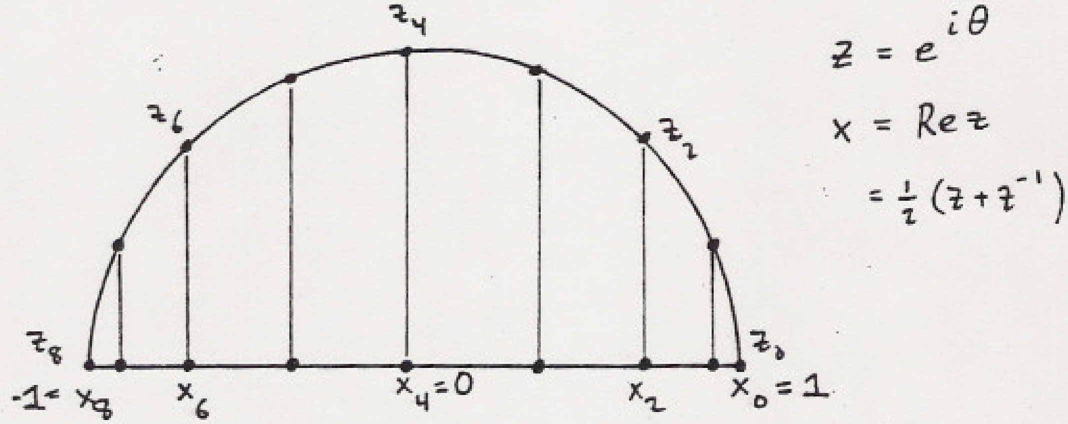


Figure 8.1.1. Chebyshev extreme points ($N=8$).

To the eye, Legendre points look much the same, although there is no elementary geometrical definition. Figure 8.1.2 illustrates the similarity:

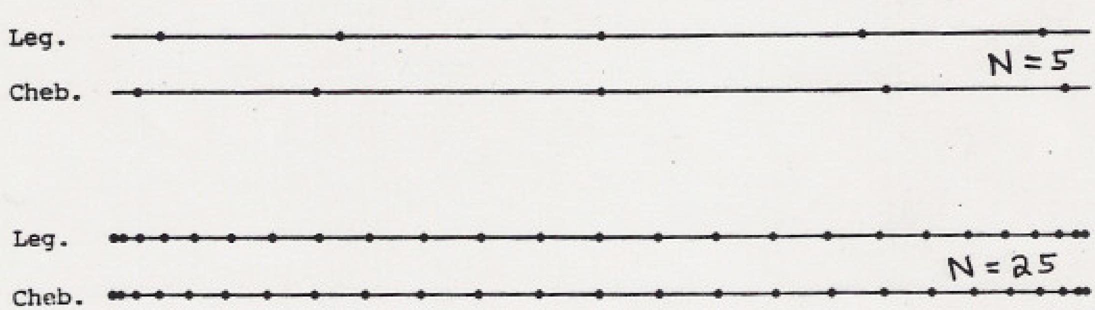


Figure 8.1.2. Legendre vs. Chebyshev zeros.

As $N \to \infty$, equispaced points are distributed with density

$$\mu(x) = \frac{N}{2} \qquad \text{Equally spaced,} \tag{8.1.2}$$

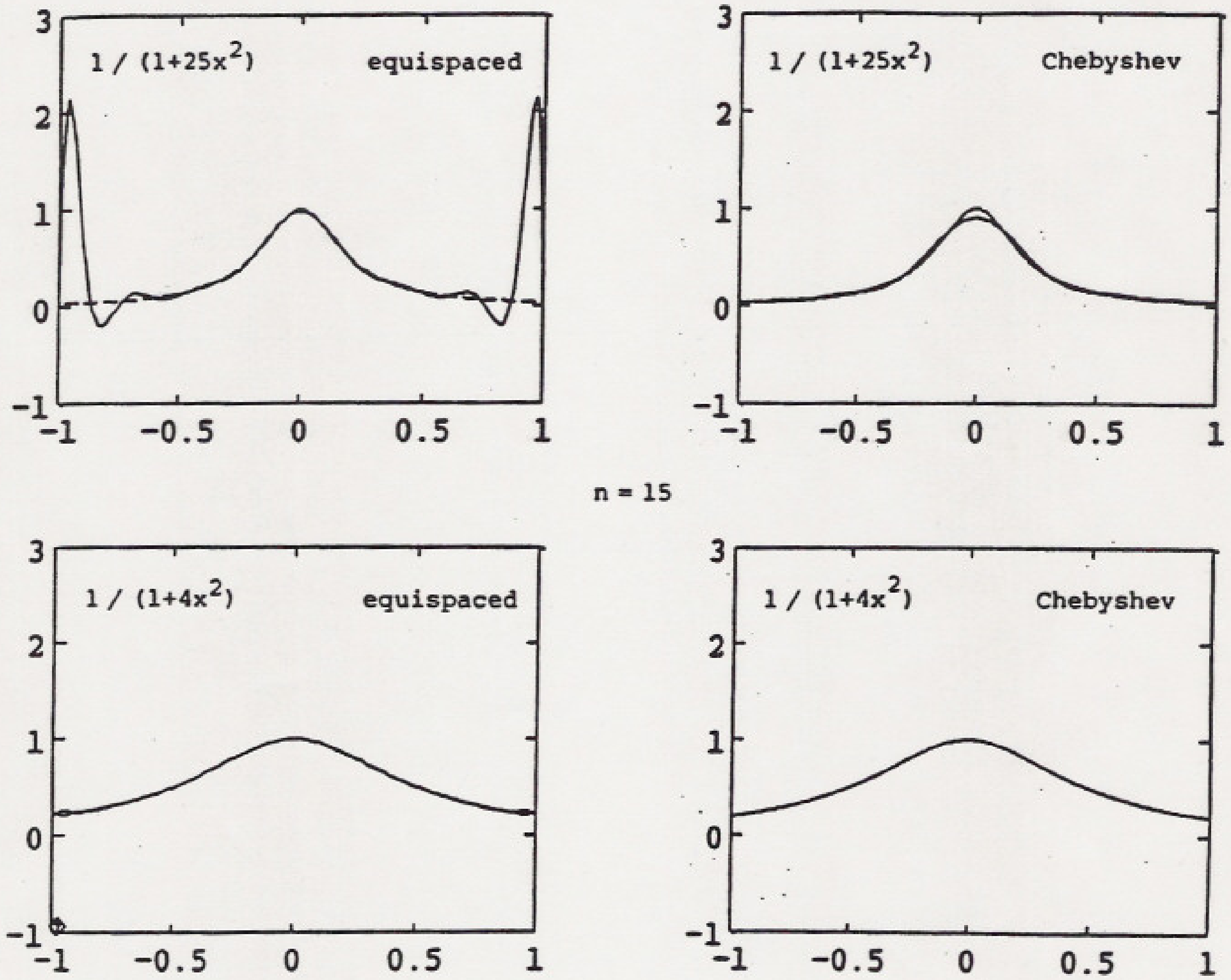
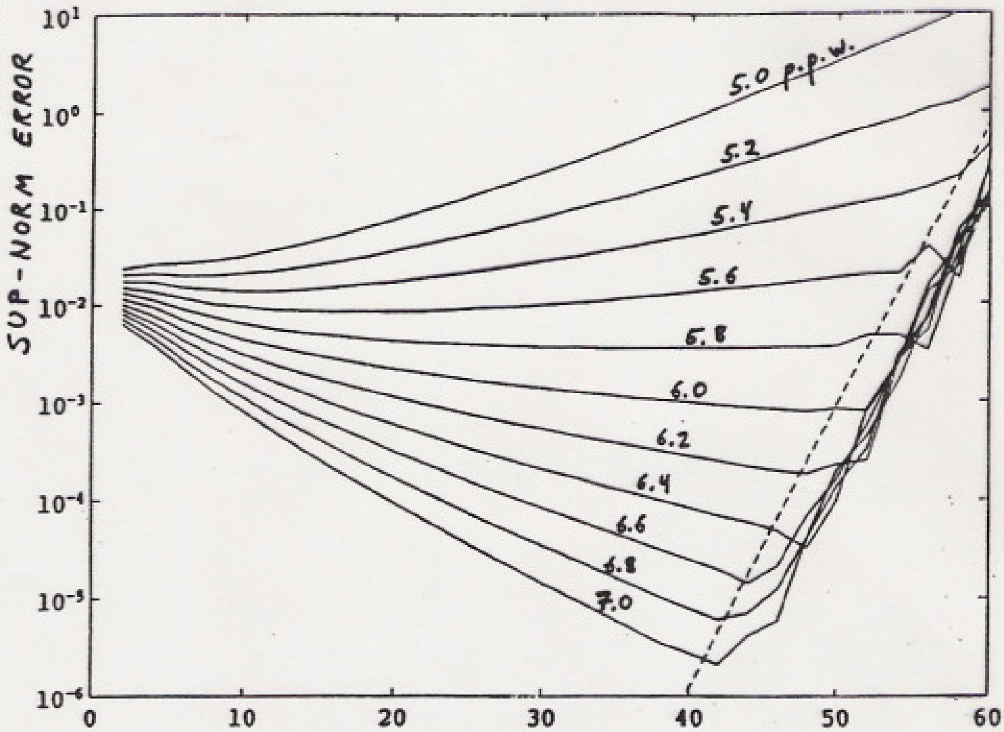
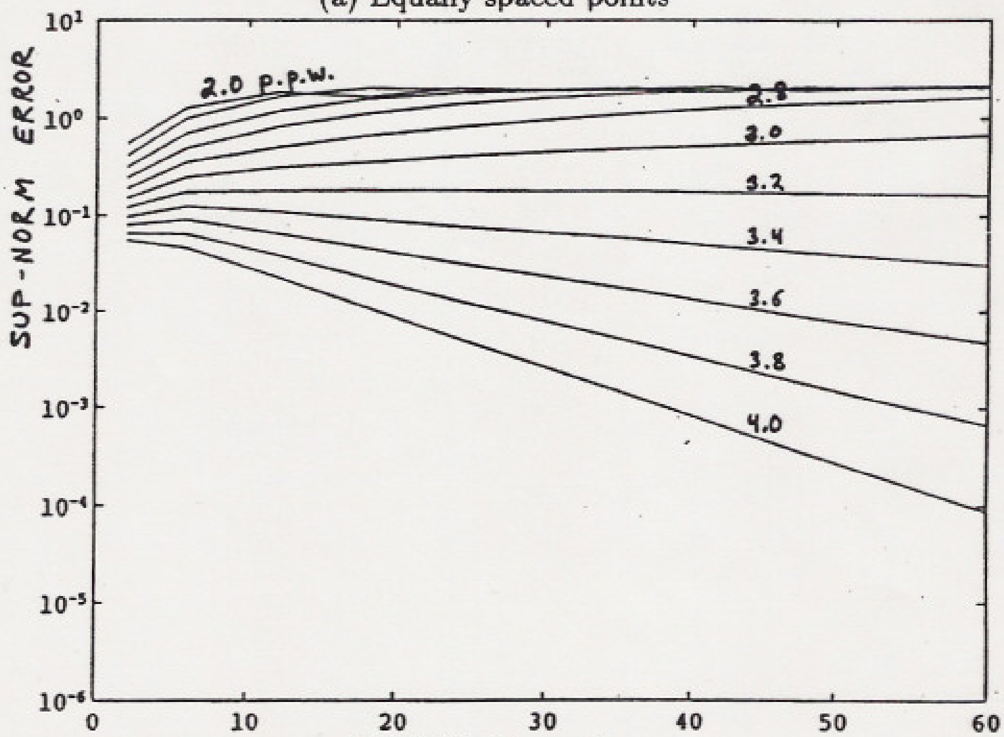


Figure 8.1.3. The Runge phenomenon.



(a) Equally spaced points



(b) Chebyshev points

Figure 8.1.5. Error as a function of N in interpolation of $\cos \alpha N x$, with α, hence the number of points per wavelength, held fixed.