Practice Problems:

4: Let $f \in C^1([x_0, x_n])$ and $x_0 < x_1 < \cdots < x_n$ be the distinct knots (nodes)

g is the cubic spline interpolant of f on the distinct knots, satisfying the end conditions $g''(x_0) = g''(x_n) = 0$.

Prove that g minimizes the quantity $\| g'' \|_{L^2} \equiv \int_{a=x_0}^{b=x_n} |g''(x)|^2 dx$.

$\Rightarrow$ basically want to show that any function G interpolating f on the knots

$$\int_a^b |G''(x)|^2 dx \geq \int_a^b |g''(x)|^2 dx$$ where $g(x)$ is the cubic spline.

define $k(x) = g(x) - G(x)$

$$\int_a^b |G''(x)|^2 dx = \int_a^b |g''(x) - k''(x)|^2 dx$$

$$= \int_a^b |g''(x)|^2 dx - 2\int_a^b g''(x) k''(x) dx + \int_a^b |k''(x)|^2 dx$$

$$\int_a^b g''(x) k''(x) dx = g''(x) k'(x) \Big|_a^b - \int_a^b g'''(x) k'(x) dx$$

$$= 0 - 0 = 0$$

$k(x_i) = g(x_i) - G(x_i) = 0$

$$\therefore \int_a^b |G''(x)|^2 dx = \int_a^b |g''(x)|^2 + \int_a^b |k''(x)|^2 dx - 0.$$

$$\geq \int_a^b |g''(x)|^2 dx$$ and the equality holds when $k''(x) = 0 \Rightarrow G(x) = g(x)$ ※

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prove that $f[x_0, x_1, \cdots, x_n] = \dfrac{f[x_1, \cdots, x_n] - f[x_0, \cdots x_{n-1}]}{x_n - x_0}$

We know that $f[x_1, \cdots, x_n] = \sum_{j=1}^{n} \dfrac{f(x_j)}{P_2'(x_j)}$

$$P_2(x) = (x - x_1)(x - x_2)\cdots(x - x_n)$$

$$f[x_0, \cdots x_{n-1}] = \sum_{j=0}^{n-1} \frac{f(x_j)}{P_1'(x_j)}$$

$$P_1(x) = (x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

$$\therefore \frac{f[x_1, \cdots x_n] - f[x_0, \cdots x_{n-1}]}{x_n - x_0}$$

$$= \frac{\sum_{j=1}^{n} \frac{f(x_j)}{P_2'(x_j)} - \sum_{j=0}^{n-1} \frac{f(x_j)}{P_1'(x_j)}}{x_n - x_0}$$

$$= \frac{\frac{f(x_n)}{P_2'(x_n)}}{x_n - x_0} + \frac{\sum_{j=1}^{n-1}\left(\frac{f(x_j)}{P_2'(x_j)} - \frac{f(x_j)}{P_1'(x_j)}\right)}{x_n - x_0} - \frac{\frac{f(x_0)}{P_1'(x_0)}}{x_n - x_0}$$

$$= \frac{f(x_n)}{(x_n - x_0)\cdots(x_n - x_{n-1})} + \frac{f(x_0)}{(x_0 - x_n)\cdots(x_0 - x_1)} + \frac{1}{x_n - x_0}\sum_{j=1}^{n-1} \frac{f(x_j)}{P_2'(x_j)} - \frac{f(x_j)}{P_1'(x_j)}$$

$$= \sum_{j=0}^{n} \frac{f(x_j)}{\Psi_n'(x_j)} \qquad \Psi_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$$

$$\therefore \frac{f[x_1, \cdots x_n] - f[x_0, \cdots, x_{n-1}]}{x_n - x_0} = f[x_0, x_1, \cdots x_n] \quad \#$$

From this we can also prove eqs. (3.2.8)

Example: Find the cubic spline that satisfies the conditions

$$S(0)=0 \qquad S(1)=1 \qquad S(2)=2 \qquad S'(0)=0 \qquad S''(2)=2$$

$$\Longrightarrow \quad x_1=0 \qquad x_2=1 \qquad x_3=2$$

$$S''(x_1)=M_1 \qquad S''(x_2)=M_2 \qquad S''(x_3)=2$$

$$S(x_1)=0=y_1, \quad S(x_2)=1=y_2, \quad S(x_3)=2=y_3$$

$\boxed{0 \le x \le 1}$

$$S(x) = \frac{1}{6}\left[(1-x)^3 M_1 + x^3 \cdot M_2\right] + x - \frac{1}{6}\left[(1-x)M_1 + xM_2\right]$$

$$S'(x) = -\frac{1}{2}(1-x)^2 M_1 + \frac{1}{2}x^2 M_2 + 1 + \frac{1}{6}M_1 - \frac{1}{6}M_2$$

$$S'(0)=0 \quad , \quad 2M_1 + M_2 - 6 = 0$$

$\boxed{1 \le x \le 2}$

$$S(x) = \frac{1}{6}\left[(2-x)^3 M_2 + 2(x-1)^3\right] + (2-x) + 2(x-1)$$
$$- \frac{1}{6}\left[(2-x)M_2 + (x-1)\cdot 2\right]$$

$$S'(x) = -\frac{1}{2}(2-x)^2 M_2 + (x-1)^2 + \frac{2}{3} + \frac{1}{6}M_2$$

Continuity of $S'(x)$ over $[a,b]$

$$\frac{x_j - x_{j-1}}{6} M_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} M_j + \frac{x_{j+1} - x_j}{6} M_{j+1}$$

$$= \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

$$\Rightarrow \frac{1}{6} M_1 + \frac{2}{3} M_2 + \frac{1}{3} = 0.$$

$$M_1 = \frac{26}{7}, \quad M_2 = -\frac{10}{7}$$

$$S(x) = \begin{cases} -\frac{6}{7}x^3 + \frac{13}{7}x^2 & 0 \le x \le 1 \\ \frac{4}{7}x^3 - \frac{17}{7}x^2 + \frac{30}{7}x - \frac{10}{7} & 1 \le x \le 2 \end{cases}$$

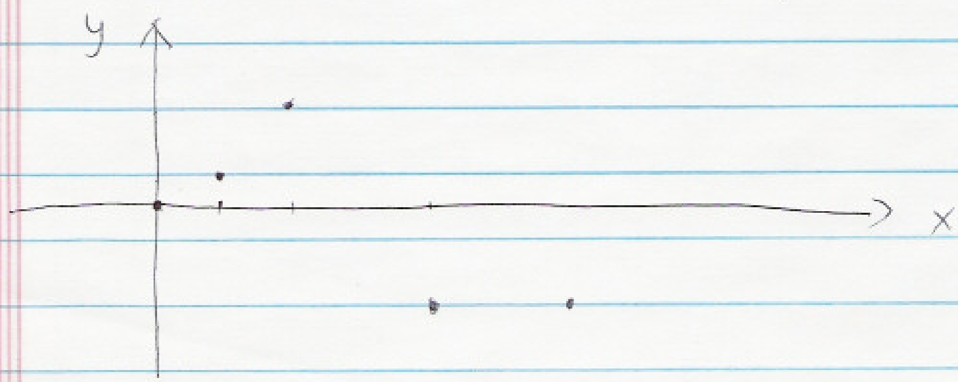
Example: Consider the data

x	0	1/2	1	2	3
y	0	1/4	1	-1	-1

ⓐ Find the piecewise linear interpolating function for the data

ⓑ Find the piecewise quadratic interpolating function.

ⓒ Find the natural cubic spline that interpolates the data.



(a)

$$\ell(x) = \frac{x}{2} \qquad 0 \le x \le \frac{1}{2}$$

$$\ell(x) = \frac{3x-1}{2} \qquad \frac{1}{2} \le x \le 1$$

$$\ell(x) = -2x+3 \qquad 1 \le x \le 2$$

$$\ell(x) = -1 \qquad 2 \le x \le 3$$

(b)

$$q(x) = ax^2+bx+c \qquad 0 \le x \le 1$$

$$\begin{cases} q(0) = 0 \\ q(\frac{1}{2}) = \frac{1}{4} \\ q(1) = 1 \end{cases} \qquad q(x) = x^2$$

$$1 \le x \le 3$$

$$q(x) = ax^2+bx+c$$

$$q(1) = 1 \qquad q(x) = x^2 - 5x + 5$$

$$q(2) = -1$$

$$q(3) = -1$$

(c).

$$\frac{x_j - x_{j-1}}{6} M_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} M_j + \frac{x_{j+1} - x_j}{6} M_{j+1}$$

$$= \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}} \qquad j = 2, 3, \cdots n-1$$

$M_1 = M_5 = 0$ in this example

$$M_1 = M_5 = 0$$

$$M_2 = M_4 = \frac{38}{7}$$

$$M_3 = \frac{-68}{7}$$

$$S(x) = \begin{cases} \frac{38}{21}x^3 + \frac{1}{21}x & 0 \le x \le \frac{1}{2} \\ -\frac{106}{21}x^3 + \frac{172}{7}x^2 - \frac{107}{21}x + \frac{6}{7} & \frac{1}{2} \le x \le 1 \\ \frac{53}{21}x^3 - \frac{87}{7}x^2 + \frac{370}{21}x - \frac{47}{7} & 1 \le x \le 2 \\ -\frac{19}{21}x^3 + \frac{57}{7}x^2 - \frac{494}{21}x + \frac{145}{7} & 2 \le x \le 3 \end{cases}$$

Recall that $S(x)$ for $x \in [x_{j-1}, x_j]$ is

$$S(x) = \frac{(x_j - x)^3 M_{j-1} + (x - x_{j-1})^3 M_j}{6\,(x_j - x_{j-1})} + \frac{(x_j - x)\,y_{j-1} + (x - x_{j-1})\,y_j}{x_j - x_{j-1}}$$

$$- \frac{1}{6}(x_j - x_{j-1})\left[(x_j - x)M_{j-1} + (x - x_{j-1})M_j\right]$$

Practice problems b

(3) $x^{(k+1)} = M\,x^{(k)} + b \qquad \|M\| < 1$

(a) The process is convergent to the unique solution of the linear system $x = Mx + b$.

$$x^{(k+1)} = M \cdot x^{(k)} + b$$

$$-)\quad x = M \cdot x + b$$

$$x^{(k+1)} - x = M\,(x^{(k)} - x)$$

$$\|x^{(k+1)} - x\| \le \|M\|\,\|x^{(k)} - x\|$$
$$\le \|M\|^k\,\|x^{(0)} - x\|$$

because $\|M\| < 1$

$\|x^{(k+1)} - x\| \to 0$ as $k \to \infty$

$\therefore\ x^{(k+1)} \to x$ as $k \to \infty$